4.2 Energy changes during simple harmonic motion (SHM) (1h)

4.2.1 Describe the interchange between kinetic energy and potential energy during SHM.

As a particle performs SHM there is a continual transfer of energy between kinetic and potential energy.

The further the object is from the mean, the smaller its kinetic energy and the greater is the potential energy.

When the particle is at maximum displacement, it stops for an instant and kinetic energy is zero.

When the particle passes through the mean, kinetic energy is a maximum and potential energy is zero.

4.2.2 Apply the expressions $E_k = \frac{1}{2} m\omega^2(x_0^2 - x^2)$ for the kinetic energy of a particle, $E_T = \frac{1}{2} m\omega^2 x_0^2$ for the total energy and $E_P = \frac{1}{2} m\omega^2 x^2$ for the potential energy.

An object at rest ($v = 0$) at its equilibrium position ($x = 0$) has neither kinetic nor potential energy. This reference condition is considered to have no total energy. $E_1 = 0$

Recall that the energy added to a system is equal to the work done as a force is exerted to displace a mass, according to:

$$\Delta E = W = F_{ave}\, s = (\tfrac{1}{2}\, m\, a)\, s$$

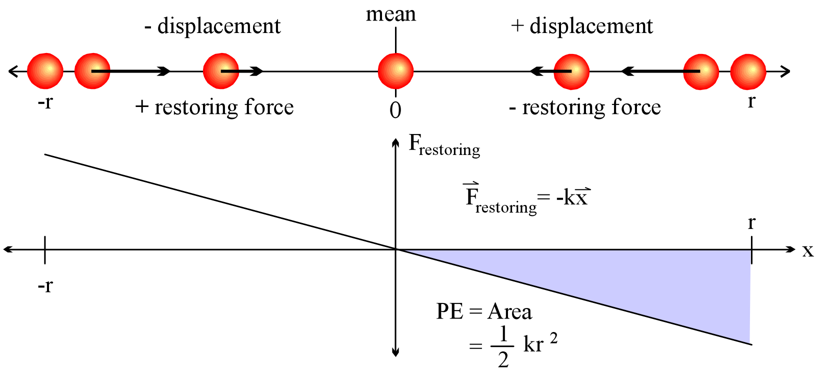
where m is the mass of the displaced object, a is its acceleration and s is the displacement from its original position.

The energy of the system after the work was done E_2 is equal to the change in energy. $E_2 = \Delta E - E_1 = \Delta E$

The **total energy** added to a system performing SHM, a is its angular acceleration and r (or x_0) is the displacement from its equilibrium position.

$$a = -\omega^2 x \qquad \text{so} \qquad E = \tfrac{1}{2}\, m\, (-\omega^2 x_0)\, x_0 \;\rightarrow\; \boxed{E = -\tfrac{1}{2}\, m\omega^2 x_0^2}$$

The diagram shows a ball performing SHM. Since the acceleration of the particle is proportional to the displacement, the restoring force must also be directly proportional to the displacement.



Potential Energy at any point *x* (using elastic energy)

$$E_P = \tfrac{1}{2}\, k\, x^2 \quad \text{but} \quad F = -k\, x \rightarrow k = -\frac{F}{x} \;\text{and}\; F = -m\omega^2 x \quad \text{so} \quad k = -\frac{(-m\omega^2 x)}{x} = +m\omega^2$$

$$\boxed{E_P = \tfrac{1}{2}\, m\, \omega^2\, x^2}$$

Kinetic Energy at any point *x* (using velocity from 4.1)

$$E_K = \tfrac{1}{2}\, m\, v^2 \quad \text{but} \quad v = \pm\omega\sqrt{x_0^2 - x^2} \quad \text{so} \quad E_k = \tfrac{1}{2}\, m(\pm\omega\sqrt{x_0^2 - x^2})^2 = \tfrac{1}{2}\, m\omega(x_0^2 - x^2)$$

$$\boxed{E_k = \tfrac{1}{2}\, m\omega(x_0^2 - x^2)}$$

Total Energy at any point *x* (combining E_p and E_k equations)

$$E_T = E_p + E_k = [\tfrac{1}{2}\, m\, \omega^2\, x^2] + [\tfrac{1}{2}\, m\, \omega^2 (r^2 - x^2)] = (\tfrac{1}{2}\, m\, \omega^2)(x^2 + r^2 - x^2 \qquad \boxed{E_T = -\tfrac{1}{2}\, m\omega^2 x_0^2}$$

The "spring constant" *k*, the period *T*, and the angular velocity *ω*

$$k = m\omega^2 = m\left(\frac{2\pi}{T}\right)^2 \text{ therefore the period } T \text{ can be expressed as } T = 2\pi\sqrt{\frac{m}{k}} \text{ and angular frequency } \omega = \sqrt{\frac{k}{m}}$$

4.2.3 Solve problems, both graphically and by calculation, involving energy changes during SHM.

A particle with mass 0.500 kg is performing SHM with an amplitude of 50.0 cm and a period of 10.0 s. At $t = 0$ it passes through its mean position and is moving in the positive direction. Calculate its speed when its displacement is 38.5 cm. (Put your calculator in radian mode!)

1. Known Values/Conditions m = 0.500 kg A = r = x_0 =50.0 cm = 0.500 m SIN $\varphi = 0$
2. Angular Frequency $\boldsymbol{\omega = 2\pi/T}$ $= 2\pi/10.0\ s^{-1}$ $= 0.2\ \pi\ s^{-1}$ $= 0.628\ rad\ s^{-1}$
3. Total Energy (E_p at r) $\boldsymbol{E_T = ½\, m\, \omega^2 x_0^2}$ = ½ 0.500 kg $(0.2\ \pi\ s^{-1})^2$ $(0.500\ m)^2$ = 0.0247 J
4. Potential (x = 0.385m) $\boldsymbol{E_P = ½\, m\omega^2 x^2}$ = ½ 0.500 kg $(0.2\ \pi\ s^{-1})^2$ $(0.385\ m)^2$ = 0.0147J
5. Kinetic (x = 0.385m) $\boldsymbol{E_K = E_T - E_P}$ = 0.0393 J – 0.0233 J = 0.0100 J

   or $E_k = \frac{1}{2} m\omega(x_0^2 - x^2) = \frac{1}{2}(0.5)(0.2\pi)^2(0.5^2 - 0.385^2) = 0.05\pi(0.250 - 0.148)$ = 0.0100 J
6. Speed (x = 0.385m) $\boldsymbol{v = \sqrt{(2E_k/m)}}$ $= (2 \cdot 0.0100J/0.500\ kg)^{1/2} = \sqrt{0.04}$ $= \pm 0.200\ m\ s^{-1}$

This agrees with the 20 cm s^{-1} that appears in the table on the previous page.

The formula $v = \pm\omega\sqrt{x_0^2 - x^2}$ produces the same result.

The graphs below show the kinetic energy, potential energy and total energy versus displacement for an object performing SHM.

The KE and PE graphs have parabolic shapes due to the x^2 term that appears in the relationships developed above. The total energy remains constant. As previously, these graphs can be plotted in Excel.