# CfE Advanced Higher Physics – Unit 2 – Waves

## SIMPLE HARMONIC MOTION

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3. Solutions of the SHM equation
4. Kinetic and potential energy in SHM

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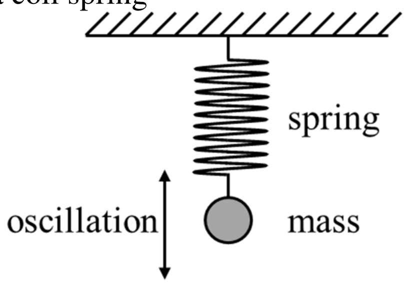
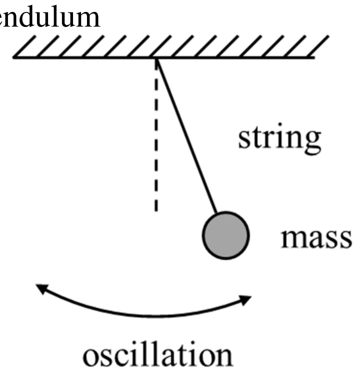
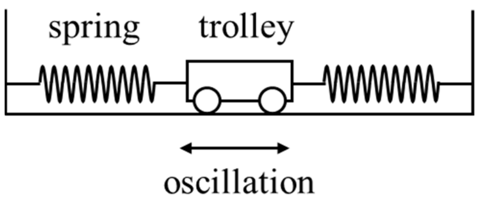
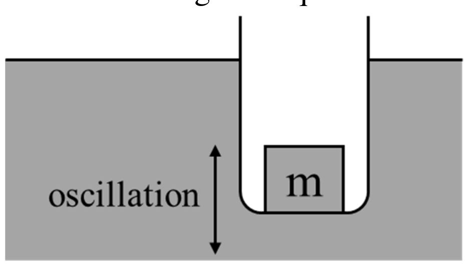
# Simple Harmonic Motion (SHM)

If an object is subject to a linear restoring force, it performs an oscillatory motion termed ‘simple harmonic’. Before a system can perform oscillations it must have (1) a means of storing potential energy and (2) some mass which allows it to possess kinetic energy. In the oscillating process, energy is continuously transformed between potential and kinetic energy.

**Note:** any motion which is periodic and complex (i.e. not simple!) can be analysed into its simple harmonic components (Fourier Analysis). An example of a complex waveform may be a sound wave from a musical instrument.

Examples of SHM

| Example and Diagram | $E_p$ stored as: | $E_k$ possesed by moving: |
|---|---|---|
| Mass on a coil spring   | elastic energy of spring | mass on spring |
| Simple pendulum   | potential energy (gravitational) of bob | mass of the bob |
| Trolley tethered between springs   | elastic energy of the springs | mass of the trolley |
| Weighted tube floating in a liquid   | potential energy (gravitational) of the tube | mass of the tube |

Note that for the mass oscillating on the spring, there is always an **unbalanced** force acting on the mass and this force is always **opposite** to the direction of its displacement. The unbalanced force is momentarily zero as the mass passes through the central position, at which it would be at rest had no initial change been made to its state. This is known as the rest position.

To see this, consider the following: when the mass is moving upwards beyond the rest position, the gravitational force (**downwards**) is greater than the spring force. Similarly when moving downwards past the rest position, the spring force (**upwards**) is greater than the gravitational force downwards.

This situation is common to all SHMs. The force which keeps the motion going is therefore called the **restoring** force.

## Definition of Simple Harmonic Motion

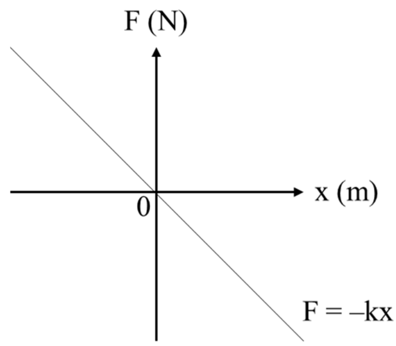
When an object is displaced from its equilibrium or rest position, and the unbalanced force is proportional to the displacement of the object and acts in the opposite direction, the motion is said to be simple harmonic.

Graph of Force against displacement for SHM

$$F = -kx$$

F is the restoring force (N)
k is the force constant ($N\ m^{-1}$)
x is the displacement (m)

The negative sign shows the direction of vector F is always opposite to vector x.




If we apply Newton's Second Law in this situation the following alternative definition in terms of acceleration, as opposed to force, is produced.

$$F = ma = m\frac{d^2x}{dt^2} = -kx$$

$$a = -\frac{k}{m}x \qquad \text{thus} \qquad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

Remember that k is a force constant which relates to the oscillating system.

The constant, $\frac{k}{m}$ is related to the period of the motion by $\omega^2 = \frac{k}{m}$, $\omega = \frac{2\pi}{T}$.

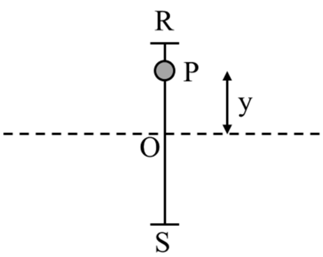
This analysis could equally well have been done using the y co-ordinate.

Thus an equivalent expression would be .

$$\frac{d^2y}{dt^2} = -\omega^2 y \qquad \frac{d^2y}{dt^2} + \omega^2 y = 0 \qquad \text{and} \qquad F = -ky$$

# Kinematics of SHM

Object P is oscillating with SHM between two fixed points R and S. The amplitude of the oscillation is described as half the distance RS, therefore ½ RS and this is given the symbol A. The displacement y is the vector OP.




O is the origin, position of zero displacement

P is the instantaneous position at displacement y

Upwards is considered the positive direction.

The period, T, of the motion is the time taken to complete one full oscillation, e.g. path O → R → O → S → O.

The frequency, f, is the number of oscillations in one second.

$$f = \frac{1}{T} \qquad \text{and because} \qquad \omega = \frac{2\pi}{T} \qquad \omega = 2\pi f$$

## Solutions of Equation for SHM

The equation $\frac{d^2y}{dt^2} = -\omega^2 y$ could be solved using integration to obtain equations for velocity, v, and displacement, y, of the particle at a particular time, t. However, the calculus involves integration which is not straightforward. We will therefore start with the solutions and use differentiation.

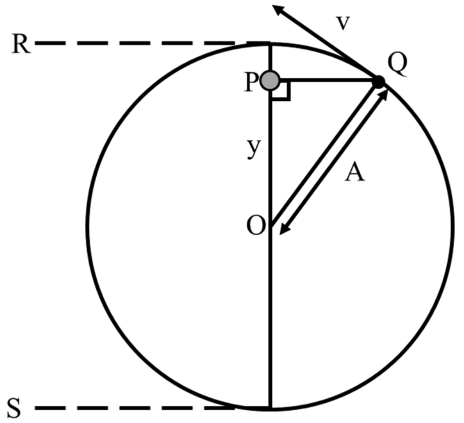
The possible solutions for the displacement, y, at time, t, depend on the **initial conditions** and are given by:

| **Displacement** $y = 0$ at $t = 0$ | **Displacement** $y = A$ at $t = 0$ |
|---|---|
| $\mathbf{y = A \sin \omega t}$ | $\mathbf{y = A \cos \omega t}$ |
| **Velocity** | **Velocity** |
| $v = \frac{dy}{dt} = \frac{d}{dt}(A \sin \omega t)$ | $v = \frac{dy}{dt} = \frac{d}{dt}(A \cos \omega t)$ |
| $\mathbf{v = A\, \omega \cos \omega t}$ | $\mathbf{v = -A\, \omega \sin \omega t}$ |
| **Acceleration** | **Acceleration** |
| $a = \frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{d}{dt}(A\, \omega \cos \omega t)$ | $a = \frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{d}{dt}(-A\, \omega \sin \omega t)$ |
| $a = -\omega^2 A \sin \omega t$ | $a = -\omega^2 A \cos \omega t$ |
| $\mathbf{a = -\omega^2 y}$ | $\mathbf{a = -\omega^2 y}$ |
| $v^2 = A^2 \omega^2 \cos^2 \omega t \qquad y^2 = A^2 \sin^2 \omega t$ | $v^2 = A^2 \omega^2 \sin^2 \omega t \qquad y^2 = A^2 \cos^2 \omega t$ |
| $\sin^2 \omega t + \cos^2 \omega t = 1$ | $\sin^2 \omega t + \cos^2 \omega t = 1$ |
| $\frac{v^2}{\omega^2 A^2} + \frac{y^2}{A^2} = 1$ | $\frac{v^2}{\omega^2 A^2} + \frac{y^2}{A^2} = 1$ |
| $v^2 = \omega^2 (A^2 - y^2)$ | $v^2 = \omega^2 (A^2 - y^2)$ |
| $\mathbf{v = \pm\, \omega \sqrt{A^2 - y^2}}$ | $\mathbf{v = \pm\, \omega \sqrt{A^2 - y^2}}$ |

## Linking SHM with Circular Motion

This allows us to examine the mathematics of the motion and is provided for interest.

If the point Q is moving with constant linear speed, v, in a circle, its projection point P on the y axis will have **displacement** $y = A\cos\theta$




radius OQ sweeps out $\omega$ rad $s^{-1}$

positive direction of y is upwards

note that $\sin\theta = \frac{QP}{OQ}$

$\sin\theta = \frac{\sqrt{A^2-y^2}}{A}$

(Viewed from the side this motion will appear identical to the one dimensional motion described on the previous page)

The **velocity** of point P is: $v_P = \frac{dy}{dt} = \frac{d}{dt}(A\cos\theta)$ and $\theta = \omega t$

$v_P = -A\omega\sin\omega t$ (negative sign: P moving down)

***Special cases***: when y = 0, $\theta = \frac{\pi}{2}$ and $\sin\theta = 1$

$v = \pm A\omega$ occurs as P goes through the origin in either direction.

when y = ± A, $\theta = 0$ or $\pi$ and $\sin\theta = 0$

$v_{min} = 0$ occurs as P reaches the extremities of the motion.

The **acceleration** of point P is: $a_P = \frac{d^2y}{dt^2} = \frac{dv_P}{dt} = \frac{d}{dt}(-A\omega\sin\omega t)$

$a_P = -A\omega^2\cos\omega t$

***Special cases***: when y = 0, $\theta = \frac{\pi}{2}$ and $\cos\theta = 0$

$a_{min} = 0$ occurs as P goes through the origin in either direction.

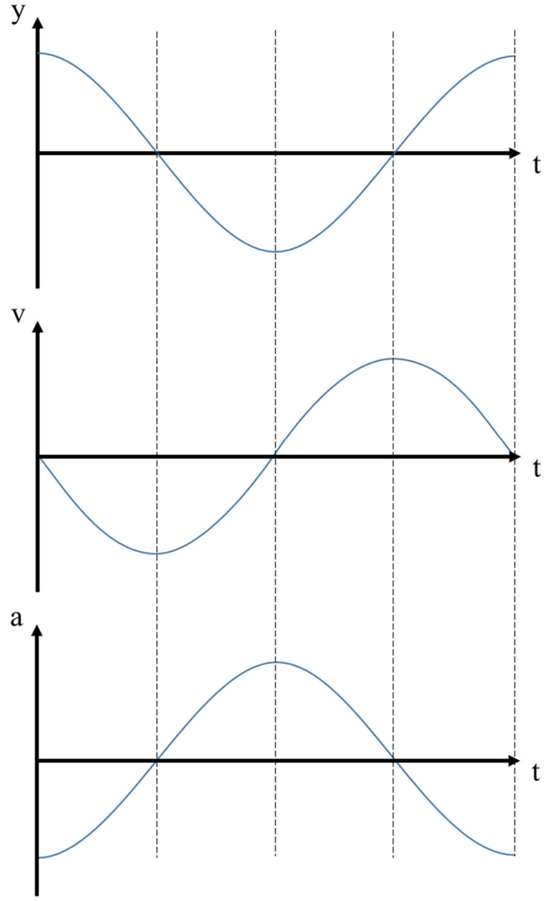
when y = ± A, $\theta = 0$ or $\pi$ and $\cos\theta = 1$

$a_{max} = \pm A\omega^2$ occurs as P reaches extremities of the motion.

**Note:** the acceleration is negative when the displacement, y, is positive and vice versa; i.e. they are out of phase, see graphs of motion which follow. Knowledge of the positions where the particle has maximum and minimum acceleration and velocity **is required**

To understand these graphs it is helpful if you see such graphs being generated using a motion sensor. In particular, pay close attention to the phases of the graphs of the motion and note that the basic shape is that of the sine/cosine graphs.

**Graphs of displacement, velocity and acceleration**




**Summary of Equations**

$$y = A\cos\omega t$$

$$v = \pm A\omega \sin\omega t$$

$$v = \pm\omega\sqrt{A^2 - y^2}$$

$$a = -A\omega^2 \cos\omega t$$

$$a = -\omega^2 y$$

Note that this form, acceleration = - $\omega^2$ y, is consistent with our definition of SHM, $\omega^2$ is a positive constant. This implies that the sine and cosine equations must be solutions of the motion.

Compare this constantly **changing** acceleration with a situation where only **uniform** acceleration was considered.

The equation used in a particular situation **depends on the initial conditions.**

| | | |
|---|---|---|
| Thus: | if $y = 0$ at time $t = 0$ | use $y = A\sin\omega t$ |
| | if $y = a$ at time $t = 0$ | use $y = A\cos\omega t$ |

Another possible solution for SHM is: $y = A\sin(\omega t + \phi)$ where $\phi$ is known as the phase angle.

***Example***

*An object is vibrating with simple harmonic motion of amplitude 0.02 m and frequency 5.0 Hz. Assume that the displacement of the object, y = 0 at time, t = 0 and that it starts moving in the positive y-direction.*

*(a) Calculate the maximum values of velocity and acceleration of the object.*

*(b) Calculate the velocity and acceleration of the object when the displacement is 0.008 m.*

*(c) Find the time taken for the object to move from the equilibrium position to a displacement of 0.012 m.*

**Solution**

Initial conditions require; $y = A \sin \omega t$; $v = A\omega \cos \omega t$; and $acc = -\omega^2 y$

$f = 5$ Hz $\omega = 2\pi f = 31.4$ rad s$^{-1}$

(a) $v_{max} = \omega A = 31.4 \times 0.02 = 0.63$ m s$^{-1}$

$acc_{max} = -\omega^2 A = -(31.4)^2 \times 0.02 = -19.7$ m s$^{-2}$

(b) $v = \pm \omega \sqrt{A^2 - y^2} = \pm\ 31.4 \sqrt{0.02^2 - 0.008^2} = \pm 0.58$ m s$^{-1}$

$acc = -\omega^2 y = -31.4^2 \times 0.008 = -7.9$ m s$^{-2}$

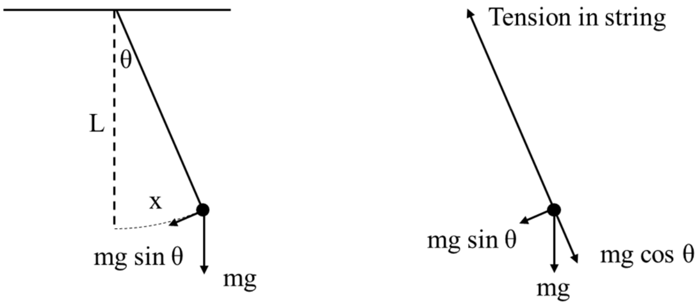
(c) use $y = A \sin \omega t$ ; $0.012 = 0.02 \sin 31.4t$ (when $y = 0.012$ m)

$\sin 31.4\,t = \frac{0.012}{0.02} = 0.6$ giving $31.4\,t = 0.644$ and $t = \frac{0.644}{31.4}$

Thus $t = 0.0205$ s (Remember that angles are in radians)

Proof that the Motion of a Simple Pendulum approximates to SHM

The sketches below show a simple pendulum comprising a point mass, m, at the end of an inextensible string of length, L. The string has negligible mass.




The restoring force F on the bob is $F = -mg \sin\theta$

If the **angle θ is small** (less than about 10°) then $\sin \theta = \theta$ in radians and $\theta = \frac{x}{L}$

Then $F = -mg\,\theta = -mg\frac{x}{L}$ Thus $F = = -mg\frac{x}{L}$

The restoring force therefore satisfies the conditions for SHM for small displacements.

Acceleration $a = -\frac{g}{L}x$ and $a = -\omega^2 x$ giving $\omega^2 = \frac{g}{L}$ ($\omega = 2\pi f$)

$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$ and the period is given by $\boxed{T = 2\pi\sqrt{\frac{L}{g}}}$

## Energy Equations for SHM

Consider a particle moving with simple harmonic motion.
The particle has maximum amplitude A and period $T = \frac{2\pi}{\omega}$

Kinetic energy equation for the particle

$$E_K = \frac{1}{2}mv^2 = \frac{1}{2}m\left[\pm\omega\sqrt{A^2\text{-}y^2}\right]^2$$

$$E_K = \frac{1}{2}m\,\omega^2(A^2\text{-}y^2)$$

Potential energy equation for the particle

When at position O the potential energy is zero, (with reference to the equilibrium position) and the **kinetic energy is a maximum**.

The kinetic energy is a maximum when y = 0: $E_{K_{\max}} = \frac{1}{2}m\omega^2A^2$

At point O total energy $E_{tot} = E_K + E_P = \frac{1}{2}m\omega^2A^2 + 0$

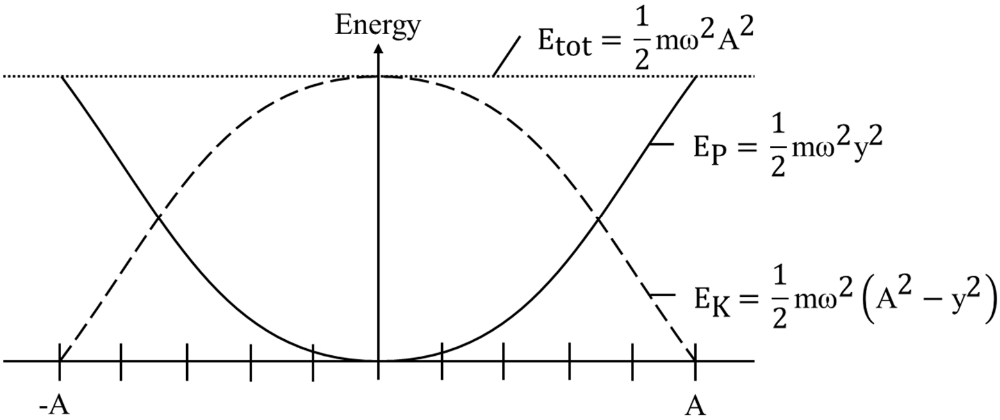
$$E_{tot} = \frac{1}{2}m\omega^2A^2 \quad \text{or} \quad E_{tot} = \frac{1}{2}k\,A^2 \quad \text{since} \quad \omega^2 = \frac{k}{m}$$

The **total energy E** is the **same at all points** in the motion.
Thus for any point on the swing: as above $E_{tot} = E_k + E_p$

$$\frac{1}{2}m\omega^2A^2 = \frac{1}{2}m\omega^2(A^2\text{-}y^2) + E_P$$

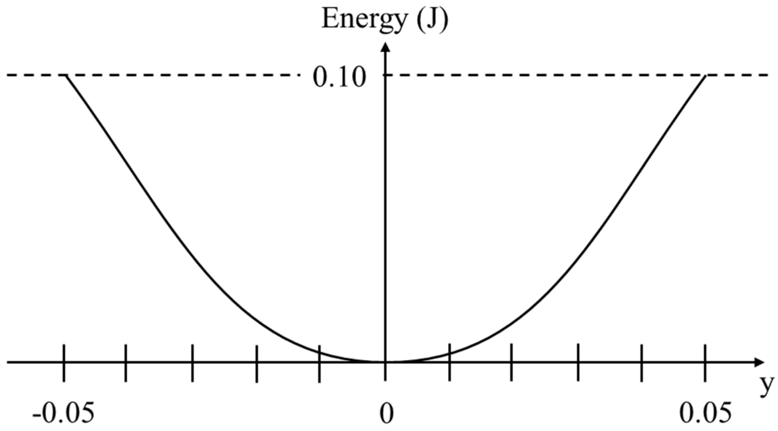
$$E_P = \frac{1}{2}m\omega^2y^2$$

The graph below shows the relation between potential energy, $E_p$, kinetic energy $E_k$, and the total energy of a particle during SHM as amplitude y changes from - A to + A.



Example on energy and SHM

*The graph below shows how the potential energy, $E_p$, of an object undergoing SHM, varies with its displacement, y. The object has mass 0.40 kg and a maximum amplitude of 0.05 m.*




*(a) (i) Find the potential energy of the object when it has a displacement of 0.02 m.*
*(ii) Calculate the force constant, k for the oscillating system (k has unit N $m^{-1}$).*

*(b) Find the amplitude at which the potential energy equals the kinetic energy.*

**Solution**

(a) (i) *From graph* $E_p = 0.10$ J

(ii) $E_p = \frac{1}{2} k y^2$

$0.1 = \frac{1}{2} k (0.02)^2$

$k = \frac{0.2}{(0.02)^2} = 500 \text{ N m}^{-1}$

(b) $E_p = E_k$

$\frac{1}{2} k y^2 = \frac{1}{2} m \omega^2 ( A^2 - y^2 )$

$= \frac{1}{2} k (A^2 - y^2)$ since $\omega^2 = \frac{k}{m}$

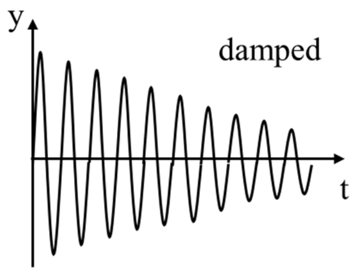
$y^2 = A^2 - y^2$ or $2 y^2 = A^2$

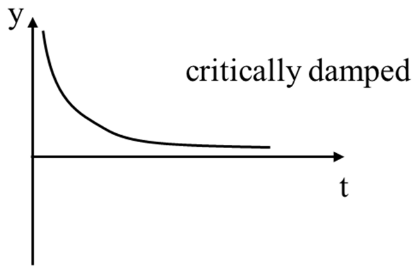
$y = \frac{A}{\sqrt{2}}$ when $E_p = E_k$

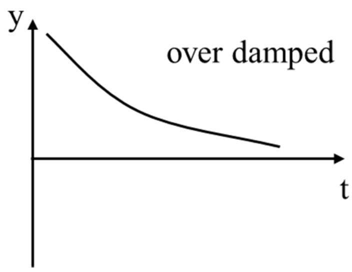
$y = \frac{0.05}{\sqrt{2}} = 0.035$ m

## Damping of Oscillations

Oscillating systems, a mass on a spring, a simple pendulum, a bobbing mass in water, all come to rest eventually. We say that their motion is **damped**. This means that the amplitude of the motion decreases to zero because energy is transformed from the system. A simple pendulum takes a long time to come to rest because the frictional effect supplied by air resistance is small - we say that the pendulum is lightly damped. A tube oscillating in water comes to rest very quickly because the friction between the container and the water is much greater - we say that the tube is heavily damped.




If the damping of a system is increased there will be a value of the frictional resistance which is just sufficient to prevent any oscillation past the rest position - we say the system is **critically damped**.

Systems which have a very large resistance, produce no oscillations and take a long time to come to rest are said to be **over damped**. In some systems over damping could mean that a system takes longer to come to rest than if underdamped and allowed to oscillate a few times.

An example of damped oscillations is a car shock absorber which has a very thick oil in the dampers. When the car goes over a bump, the car does not continue to bounce for long. Ideally the system should be critically damped. As the shock absorbers get worn out the bouncing may persist for longer.

The graphs below give a graphical representation of these different types of damping.

# Waves

## Wave Motion

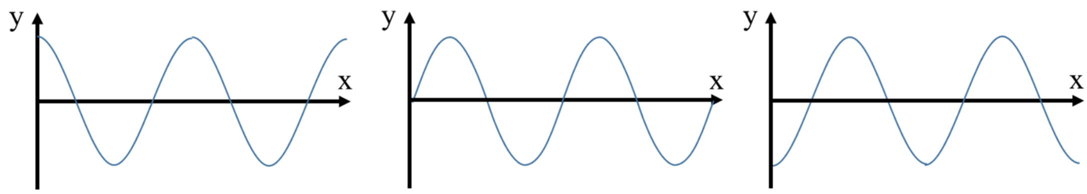
In a wave motion energy is transferred from one position to another with no net transport of mass.

Consider a water wave where the movement of each water particle is at right angles (transverse) to the direction of travel of the wave. During the wave motion each particle, labelled by its position on the x-axis, is displaced some distance y in the transverse direction. In this case, "no net transfer of mass" means that the water molecules themselves do not travel with the wave - the wave energy passes over the surface of the water, and in the absence of a wind/tide any object on the surface will simply bob up and down.

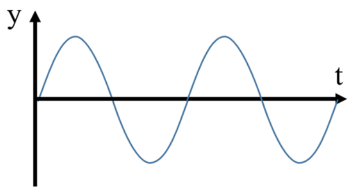
## The Travelling Wave Equation

The value of the displacement, y, depends on which particle of the wave is being considered. It is dependent on the x value, and also on the time, t, at which it is considered. Therefore y is a function of x and t giving $y = f(x, t)$. If this function is known for a particular wave motion we can use it to predict the position of any particle at any time.

Below are 3 'snapshots' of a transverse wave, moving left to right, taken at different times showing how the displacement of different particles varies with position x.




The following diagram shows the movement of **one** particle on the wave as a function of time.




Initial condition at the origin:
$y = 0$ when $t = 0$

For a wave travelling from **left to right** with speed v, the particle will be performing SHM in the y-direction.

The equation of motion of the particle will be:

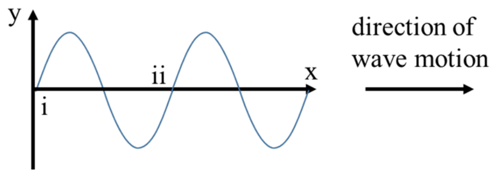
$y = A \sin \omega t$ where A is the amplitude of the motion.

The displacement of the particle is simple harmonic. The sine or cosine variation is the simplest description of a wave.

When $y = 0$ at $t = 0$ the relationship for the wave is $y = A \sin \omega t$, as seen above.
When $y = A$ at $t = 0$ the relationship for the wave is $y = A \cos \omega t$.

## Deriving the travelling wave equation

Consider a snapshot of the wave as shown below.




The time, t, for the wave disturbance to travel from (i) (x = 0) to (ii) (x = x) is $\frac{x}{v}$ .

Consider particle (i) at position x = 0. The equation of motion of particle (i) is given by y = A sin ωt, where t is the time at which the motion of particle (i) is observed.

Now consider particle (ii) at position x = x and the time t = t.
Since wave motion is a repetitive motion:

motion of particle (ii) (x = x, t = t) = motion of particle (i) (x = 0, t = $\frac{x}{v}$),

[i.e. the motion of particle (ii) = motion of particle (i) at the **earlier** time of t = $\frac{x}{v}$ ].

General motion of particle (i) is given by y = A sin ωt, but in this case t = t – $\frac{x}{v}$

hence $y = A \sin \omega(t - \frac{x}{v})$.

Motion of particle (ii) (x = x, t = t) is also given by $y = A \sin \omega(t - \frac{x}{v})$.

In general: $y = A \sin \omega(t - \frac{x}{v})$ also $\omega = 2\pi f$ and $v = f\lambda$

$y = A \sin 2\pi f(t - \frac{x}{f\lambda})$ which gives

$$y = A \sin 2\pi\left(ft - \frac{x}{\lambda}\right)$$

for a wave travelling from left to right in the **positive** x-direction.

and

$$y = A \sin 2\pi\left(ft + \frac{x}{\lambda}\right)$$

for a wave travelling from right to left in the **negative** x-direction.

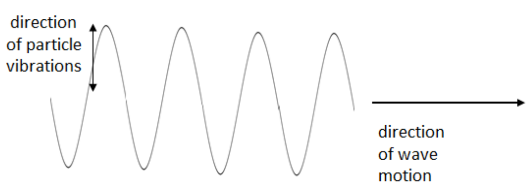
## The Intensity/Energy of a Wave

The intensity or energy of a wave is directly proportional to the square of its amplitude.

$$\text{Intensity or Energy} \propto A^2$$

# Longitudinal and transverse waves

With transverse waves, as in water waves, each particle oscillates at right angles to the direction of travel of the wave (left diagram). In longitudinal waves, such as sound waves, each particle vibrates along the direction of travel of the wave (right diagram).



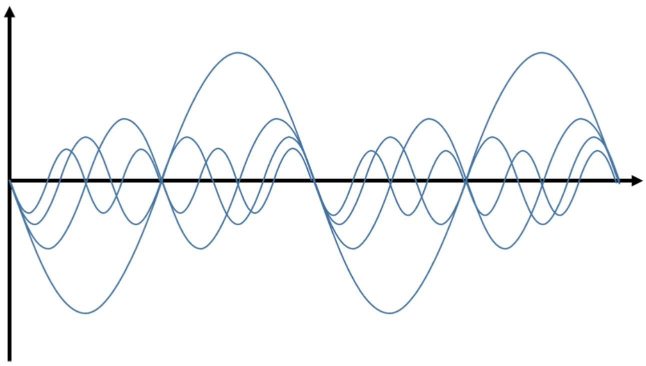

## Principle of Superposition of Waveforms

Travelling waves can pass through each other without being altered. If two stones are dropped in a calm pool, two sets of circular waves are produced. These waves pass through each other. However at any point at a particular time, the disturbance at that point is the algebraic sum of the individual disturbances. In this example, when a 'trough' from one wave meets a 'crest' from the other wave (the waves are **out of phase** at this location) the water will remain calm due to an effective cancelling out (known as destructive interference)

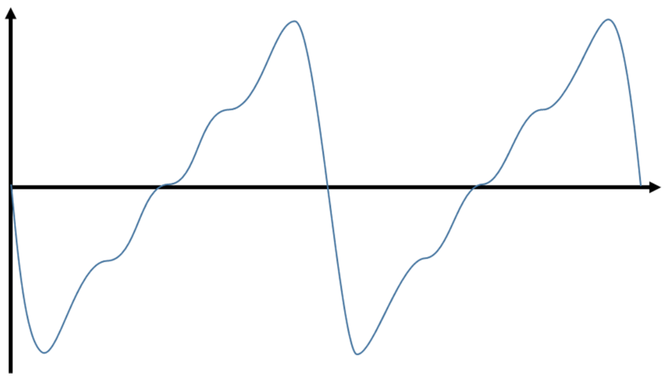
A **periodic** wave is a wave which repeats itself at regular intervals. All periodic waveforms can be described by a mathematical series of sine or cosine waves, known as a Fourier Series. For example a saw tooth wave can be expressed as a series of individual sine waves.

$$y\,(t) = -\frac{1}{\pi}\sin\omega t - \frac{1}{2\pi}\sin 2\omega t - \frac{1}{3\pi}\sin 3\omega t - \ldots$$

The graph below shows the first four terms of this expression.



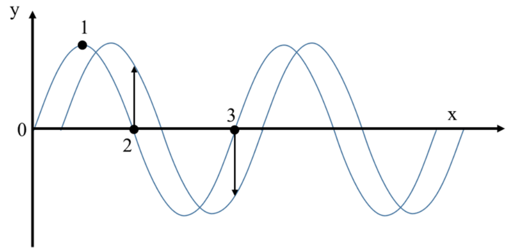
When all these terms are superimposed (added together) the graph below is obtained. Notice that this is tending to the saw tooth waveform. If more terms are included it will have a better saw tooth form.



## Phase Difference

A phase difference exists between two points on the same wave.

Consider the snapshots below of a wave travelling to the right in the positive x-direction.




Points 0 and 3 have a phase difference of $2\pi$ radians.
They are both at zero displacement and will next be moving in the negative direction.
They are separated by one wavelength ($\lambda$).

Points 0 and 2 have a phase difference of $\pi$ radians.
They both have zero displacement but 2 will next be going positive and 0 will be going negative. They are separated by $\lambda/2$. Notice that points 1 and 2 have a phase difference of $\pi/2$.

The table below summarises phase difference and separation of the points.

| Phase difference | Separation of points |
|---|---|
| 0 | 0 |
| $\pi/2$ | $\lambda/4$ |
| $\pi$ | $\lambda/2$ |
| $2\pi$ | $\lambda$ |

Notice that $\frac{\text{phase difference}}{\text{separation of points}} = \frac{2\pi}{\lambda}$ = constant.

If the phase difference between two particles is $\phi$ when the separation of the particles is x, then $\frac{\phi}{x} = \frac{2\pi}{\lambda}$.

In general, for two points on a wave separated by a distance x the phase difference is given by:

$$\phi = 2\pi\frac{x}{\lambda} \qquad \text{where } \phi \text{ is the phase angle in radians}$$

**Example**

*A travelling wave has a wavelength of 60 mm. A point P is 75 mm from the origin and a point Q is 130 mm from the origin.*
*(a) What is the phase difference between P and Q?*
*(b) Which of the following statements best describes this phase difference:*
*'almost completely out of phase'; 'roughly ¼ cycle out of phase'; 'almost in phase'.*

**Solution**

(a) separation of points = 130 - 75 = 55 mm = 0.055 m

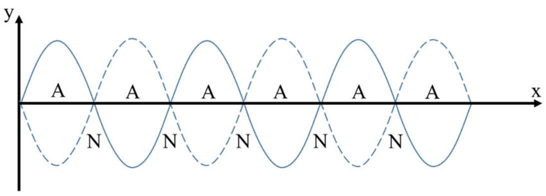
phase difference $= 2\pi\frac{0.055}{0.060} = 5.76$ radians

(b) P and Q are separated by 55 mm which is almost one wavelength, hence they are 'almost in phase'. Notice that 5.76 radians is 330°, which is close to 360°.

## Stationary Waves

A stationary wave is formed by the interference between two waves, of the **same** frequency and amplitude, travelling in **opposite** directions. For example, this can happen when sound waves are reflected from a wall and interfere with the waves approaching the wall.

A stationary wave travels neither to the right nor the left, the wave 'crests' remain at fixed positions while the particle displacements increase and decrease in unison.




A - antinodes

N - nodes

There are certain positions which always have **zero amplitude** independent of the time we observe them; these are called **nodes**.

There are other points of **maximum amplitude** which are called **antinodes**.
Note that the distance between each node and the next node is $\frac{\lambda}{2}$ and that the distance between each antinode and the next antinode is $\frac{\lambda}{2}$.

### Use of standing waves to measure wavelength

Standing waves can be used to measure the wavelength of waves. The distance across a number of minima is measured and the distance between consecutive nodes determined and the wavelength calculated. This method can be used for sound waves or microwaves.

### Formula for standing waves

Consider the two waves $y_1$ and $y_2$ travelling in the opposite direction, where

$$y_1 = A \sin 2\omega \left(ft - \frac{x}{\lambda}\right) \qquad \text{and} \qquad y_2 = A \sin 2\omega \left(ft + \frac{x}{\lambda}\right)$$

When these two waves meet the resultant displacement y is given by

$$y = y_1 + y_2 = A \sin 2\omega \left(ft - \frac{x}{\lambda}\right) + A \sin 2\omega \left(ft + \frac{x}{\lambda}\right)$$

$y = 2A \sin 2\pi f t \ \cos \frac{2\pi x}{\lambda}$ (using $b \sin P + b \cos Q = 2b \sin \frac{P+Q}{2} \cos \frac{P-Q}{2}$)

Giving $$y = 2A \sin \omega t \cos \frac{2\pi x}{\lambda}$$

Notice that the equation is a function of two trigonometric functions, one dependent on time, t, and the other on position, x. Consider the part which depends on position. We can see that there are certain fixed values of x for which $\cos \frac{2\pi x}{\lambda}$ is equal to zero. These are $x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}$, etc. This shows that there are certain positions where y = 0 which are independent of the time we observe them - the nodes.
The antinodes are therefore given by $\cos \frac{2\pi x}{\lambda} = 1$, that is at $x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}$, etc.

# Interference - Division of Amplitude

## Producing interference

Interference of waves occurs when waves overlap. There are two ways to produce an interference pattern for light: division of amplitude and division of wavefront. Both of these involve splitting the light from a single source into two beams. We will consider division of amplitude first and division of wavefront later.

Before we consider specific examples of either we need to consider some general properties of interference.

### Coherent sources

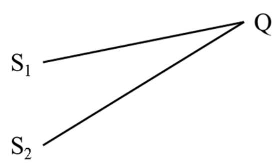
Two coherent sources must have a **constant phase difference**. Hence they will have the **same frequency**.

To produce an interference pattern for light waves the two, or more, overlapping beams always come from the **same single source**. When we try to produce an interference pattern from two separate light sources it does not work because light cannot be produced as a continuous wave. Light is produced when an electron transition takes place from a higher energy level to a lower energy level in an atom. The energy of the photon emitted is given by $\Delta E = hf$ where $\Delta E$ is the difference in the two energy levels, f is the frequency of the photon emitted and h is Planck's constant. Thus a source of light has continual changes of phase, roughly every nanosecond, as these short pulses of light are produced. Two sources of light producing the same frequency will not have a constant phase relationship so will not produce clear interference effects.

This is not the case for sound waves. We can have two separate loudspeakers, connected to the same signal generator, emitting the same frequency which will produce an interference pattern.

### Path Difference and Optical Path Difference

Sources $S_1$ and $S_2$ are two coherent sources **in air**.




The path difference is $(S_2Q - S_1Q)$. For constructive interference to take place at Q, the waves must be in phase at Q. Hence the path difference must be a whole number of wavelengths.

$$(S_2Q - S_1Q) = m\lambda \qquad \text{where } m = 0, 1, 2, 3...$$

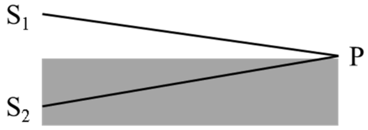
(**Note**: the letter m is used to denote an integer, not n, since we use n for refractive index.)

Similarly, for destructive interference to take place the waves must be out of phase at Q by $\lambda/2$ (a 'crest' from $S_1$ must meet a 'trough' from $S_2$).

$$(S_2Q - S_1Q) = \left(m + \tfrac{1}{2}\right)\lambda \qquad \text{where } m = 0, 1, 2, 3...$$

Optical path difference

In some situations the path followed by one light beam is inside a transparent material of refractive index, n. Consider two coherent beams $S_1$ and $S_2$ where $S_1P$ is in air and $S_2P$ is in perspex of refractive index n = 1.5. We will consider the point P itself to be in air.




The geometrical path difference $S_1P$ - $S_2P$ is zero.

But will there be constructive interference at P?

The wavelength inside the perspex is **less** than that in air $\lambda_{perspex} = \frac{\lambda_{air}}{1.5}$. Hence the waves from $S_1$ and $S_2$ **may not** arrive at P in phase. For example, if there were exactly Z whole waves between $S_1P$, there will be 1.5 x Z waves between $S_2P$ which may or may not be a whole number of wavelengths.

The **optical path length** must be considered not the geometrical path length.

| |
|---|
| Optical path length = refractive index × geometrical path length |

Thus the relationships for constructive and destructive interference must be considered for **optical path lengths**, $S_2P$ and $S_1P$.

| | | |
|---|---|---|
| For constructive interference | $(S_2P - S_1P) = m\lambda$ | where m is an integer |
| For destructive interference | $(S_2P - S_1P) = (m + \frac{1}{2})\lambda$ | where m is an integer |

Phase difference and optical path difference

The optical path difference is the difference in the two optical path lengths, namely $(S_2P - S_1P)$ in our general example.

The phase difference is related to the optical path difference:

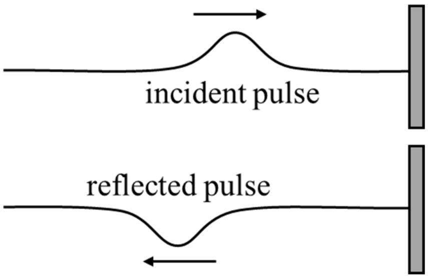
$$\text{phase difference} = \frac{2\pi}{\lambda} \times \text{optical path difference}$$

where $\lambda$ is the wavelength in vacuum.

Notice that when the optical path difference is a whole number of wavelengths, the phase difference is a multiple of $2\pi$, i.e. the waves are in phase.

## Phase Change on Reflection

To understand interference caused by multiple reflections it is necessary to consider what happens when a light wave moving in air hits a material such as glass.
On a large scale we can see what happens to the wave when a pulse on a rope or 'slinky' reflects off a dense material such as a wall.




The reflected pulse is said to undergo a phase change of 180° or π radians. The reflected pulse is 180° out of phase with the incident pulse. If these two pulses were to meet they would momentarily cancel as they passed one another.

There is a similar phase change when a light wave is reflected off a sheet of glass.

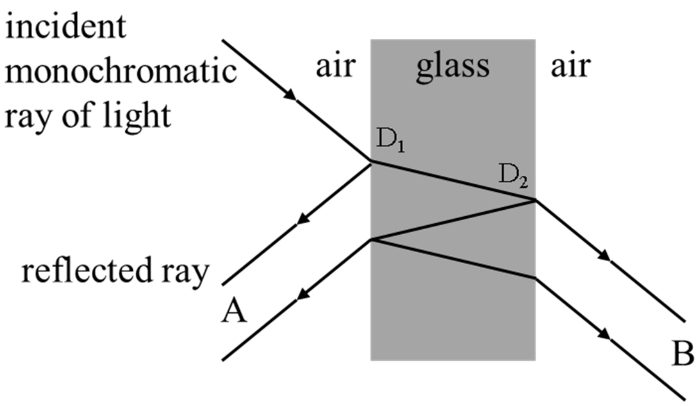
In general for **light** there is a **phase change of π on reflection** at an interface where there is an **increase** in optical density, e.g. a higher refractive index such as light going from air to glass. There is **no** phase change on reflection where there is a decrease in optical density, e.g. a lower refractive index such as light going from glass to air.

# Division of Amplitude

This involves splitting a single light beam into two beams, a reflected beam and a transmitted beam, at a surface between two media of different refractive index. In some cases multiple reflections can occur and more than two beams are produced.

## Thin parallel sided film

Interference by division of amplitude can be produced by thin films as shown below.



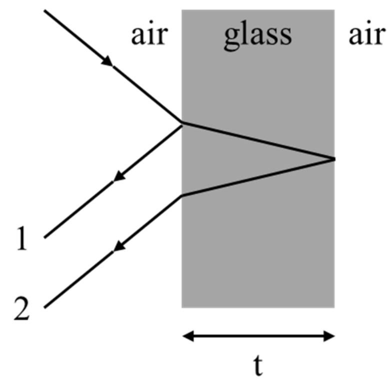

Notice that an **extended** source can be used. The amplitude of the beam is divided by reflection and transmission at $D_1$, and again by reflection and transmission at $D_2$ at the back of the glass sheet.

An eye, at A, will focus the reflected beams and an eye at B will focus the transmitted beams. Thus interference patterns can be observed in both the reflected and transmitted beams.

## Condition for maxima and minima in the fringes formed in a thin film

The following explanations are for light incident normally on a thin film or sheet of glass. The diagrams only show light paths at an angle to distinguish clearly the different paths.

### Reflected light




The ray following path 1 reflects off the glass which has a higher refractive index than air. It therefore experiences a π phase change.

The ray following path 2 reflects off air so experiences no phase change on reflection. However, it travels through the glass twice so has an optical path difference compared to ray 1 of 2nt, where n is the refractive index of the glass.

Therefore for constructive interference for the reflected light, i.e. for rays 1 and 2 to be in phase, then the optical path difference 2nt must give a π phase change. Therefore:

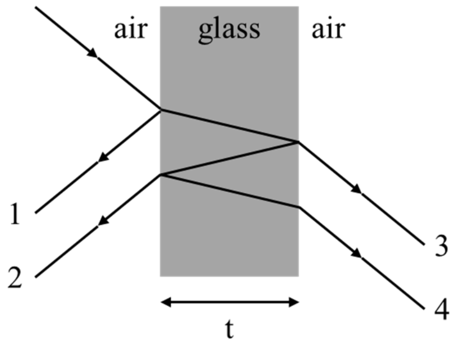
$$2nt = \left(m + \frac{1}{2}\right) \lambda \qquad \text{where m is an integer.}$$

For destructive interference for the reflected light, i.e. for rays 1 and 2 to be exactly out of phase, then the optical path difference 2*nt* must give zero phase change. Therefore:

$$2nt = m\,\lambda \qquad \text{where m is an integer.}$$

Note that these statements are the reverse of what we are used to seeing.

<u>Transmitted light</u>




The ray following path 3 passes through the glass with zero phase change.

The ray following path 4 reflects off air twice so experiences no phase changes on reflection. However, it travels through the glass twice more than path 3 so has an optical path difference compared to ray 3 of 2nt, where n is the refractive index of the glass.

Therefore for constructive interference for the transmitted light, i.e. for rays 3 and 4 to be in phase, then the optical path difference $2nt$ must give zero phase change. Therefore:

$$2nt = m\lambda \qquad \text{where m is an integer.}$$

For destructive interference for the transmitted light, i.e. for rays 3 and 4 to be exactly out of phase, then the optical path difference 2nt must give a π phase change. Therefore:

$$2nt = \left(m + \frac{1}{2}\right)\lambda \qquad \text{where m is an integer.}$$

**Note**
For a certain thickness of thin film the conditions are such that the reflected light and transmitted light have opposite types of interference. Therefore energy is conserved at all times.

**Example**
*A sheet of mica is 4.80 µm thick. Light of wavelength 512 nm is shone onto the mica. When viewed from above, will there be constructive, destructive, or partial destructive interference? The refractive index of mica is 1.60 for light of this wavelength.*

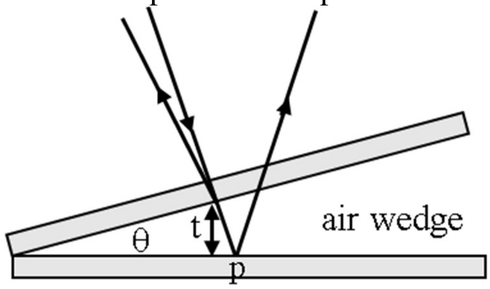
**Solution**
For destructive interference

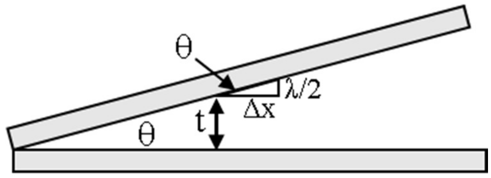
$$2nt = m\lambda$$
$$2 \times 1.60 \times 4.80 \times 10^{-6} = m \times 512 \times 10^{-9}$$
$$m = 30$$

This is an integer. Hence destructive interference is observed.

## Wedge Fringes

Two glass slides are arranged as shown below.
Division of amplitude takes place at the *lower* surface of the top glass slide.



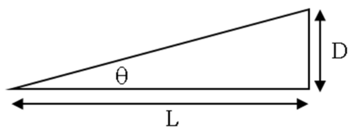

When viewed from above the optical path difference = 2t
There is a phase difference of π on reflection at p. Hence the condition for a dark fringe is $2t = m\lambda$ assuming an air wedge.

For the next dark fringe t increases by $\frac{\lambda}{2}$ (see right hand sketch above).
Thus the spacing of fringes, Δx, is such that $\tan\theta = \frac{\lambda}{2\Delta x}$

$$\Delta x = \frac{\lambda}{2\tan\theta}$$

For a wedge of length L and spacing D

$$\tan\theta = \frac{D}{L}$$




The fringe spacing is given by

$$\Delta x = \frac{\lambda L}{2D}$$

where λ is the wavelength of light in air.

In practice the distance across a number of fringes is measured and Δx determined.

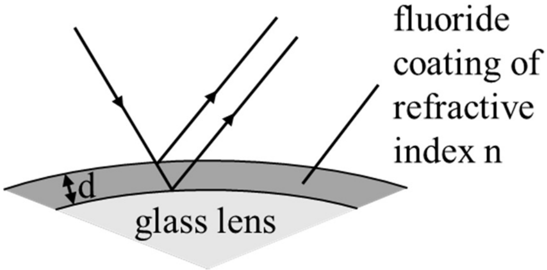
Notice that the fringes are formed inside the wedge, and that the two reflected rays are diverging. The eye, or a microscope, must be focussed between the plates for viewing the fringes.

A wedge can be formed by two microscope slides in contact at one end and separated by a human hair or ultra-thin foil at the other end. In this way the diameter of a human hair can be measured.

Similarly, if a crystal is placed at the edge and heated, the thermal expansion can be measured by counting the fringes as the pattern changes.

Non-reflecting Coating

Good quality lenses in a camera reflect very little light and appear dark or slightly purple. A thin coating of a fluoride salt such as magnesium fluoride on the surface of the lens allows the majority of the light falling on the lens to pass through.
The refractive index, n, of the coating is chosen such that $1 < n < n_{glass}$.




Notice that there is a phase change of $\pi$ at both the first and second surfaces.

For cancellation of reflected light: optical path difference $= \frac{\lambda}{2}$.

Optical path in fluoride = 2nd thus

$$2nd = \frac{\lambda}{2}$$

$$d = \frac{\lambda}{4n}$$

Complete cancellation is for one particular wavelength only. Partial cancellation occurs for other wavelengths.

The wavelength chosen for complete cancellation is in the yellow/green (i.e. middle) of the spectrum. This is why the lens may look purple because the reflected light has no yellow/green present. The red and blue light are partially reflected to produce the purple colour observed.

Colours in thin films

When a soap film is held vertically in a ring and is illuminated with monochromatic light it initially appears coloured all over. However when the soap drains downwards a wedge shaped film is produced, with the top thinner than the bottom. Thus horizontal bright and dark fringes appear. When illuminated by white light, colours are formed at positions where the thickness of the film is such that constructive interference takes place for that particular colour. Just before the soap film breaks, the top appears black because the film is so thin there is virtually no path difference in the soap. Destructive interference occurs because of the phase change on reflection.

Similar colours are observed when a thin film of oil is formed on water.

# Interference - Division of Wavefront

## Division of Wavefront

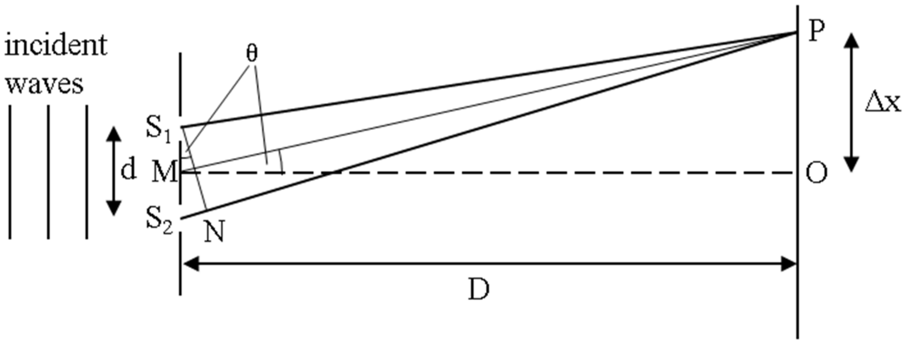
When light from a single point source is incident on two small slits, two coherent beams of light can be produced. Each slit acts as a secondary source due to diffraction.

If an extended source is used, each part of the wavefront will be incident on the slit at a different angle. Each part of the source will then produce a fringe pattern, but slightly displaced. When the intensity of all the patterns is summed the overall interference pattern may be lost. However a line source parallel to the slits is an exception.

Compare this with the use of an **extended** source in 'division of amplitude'.

## Young's Slits Experiment

The diagram below shows light from a single source of monochromatic light incident on a double slit. The light diffracts at each slit and the overlapping diffraction patterns produce interference.




A bright fringe is observed at P. Angle PMO is θ.
N is a point on $S_1P$ such that $NP = S_1P$. Since P is the $n^{th}$ bright fringe $S_2N = n\lambda$
For **small** values of θ $S_1N$ cuts MP at almost $90^0$ giving angle $S_2 S_1N = \theta$.

Again providing θ is **very small**, $\sin\theta = \tan\theta = \theta$ (in radians)

From triangle $S_2S_1N$: $\theta = \dfrac{\lambda}{d}$ also from triangle PMO: $\theta = \dfrac{\Delta x}{D}$

Thus $\dfrac{\Delta x}{D} = \dfrac{\lambda}{d}$ or $\Delta x = \dfrac{\lambda D}{d}$

Giving the fringe separation between adjacent fringes Δx

$$\Delta x = \frac{\lambda D}{d}$$

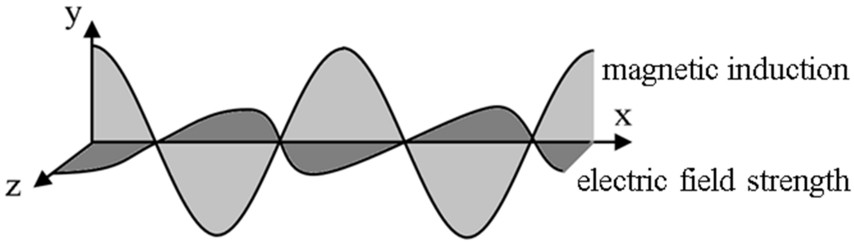
**Note**
This formula only applies if x<<D, which gives a small value for θ. This is likely to be true for light waves but not for microwaves.

The position of the fringes is dependent on the wavelength. Thus if white light is used we can expect overlapping colours either side of a central white maximum. The red part of the spectrum, with the longer wavelength, will be the furthest from this white maximum ($\Delta x_{red} > \Delta x_{violet}$ since $\lambda_{red} > \lambda_{violet}$).

# Polarisation

## Polarised and unpolarised waves

Light is a travelling wave, and is part of the electromagnetic spectrum. In all **electromagnetic waves** the electric field and magnetic field vary. The diagram below shows a 3-dimensional picture of such a wave.




The above diagram shows the variation of the electric field strength, E, in the x-z plane and the variation of the magnetic induction, B, in the x-y plane. In this example the electric field strength is only in one plane. The wave is said to be plane polarised, or **linearly polarised**. For example, in Britain this is the way that T.V. waves are transmitted. Aerials are designed and oriented to pick up the vertical electric field strength vibrations. These vibrations contain the information decoded by the electronic systems in the television.

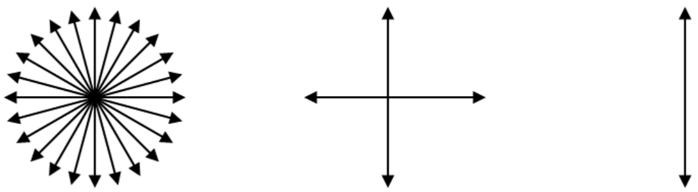
Notice that the electromagnetic wave is made up of two mutually perpendicular transverse waves. The oscillations of E and B.

Light from an ordinary filament lamp is made up of many separate electromagnetic waves produced by the random electron transitions in the atoms of the source. So unlike the directional T.V waves, light waves from a lamp consist of many random vibrations. This is called an **unpolarised** wave.

When looking at an **unpolarised** wave coming towards you the direction of the electric field strength vector would appear to be vibrating in all direction, as shown in the diagram (i) on the left below. The magnetic induction vector would be perpendicular to the electric field strength vector, hence this too would be vibrating in all directions However when discussing polarisation we refer to the electric field strength vector only.

All the individual electric field strength vectors could be resolved in two mutually perpendicular direction to give the other representation of a **unpolarised** wave, as shown below in the centre diagram (ii).

The right hand diagram (iii) represents a **polarised** wave.



(i) unpolarised (ii) unpolarised (iii) polarised

**Longitudinal and transverse waves**

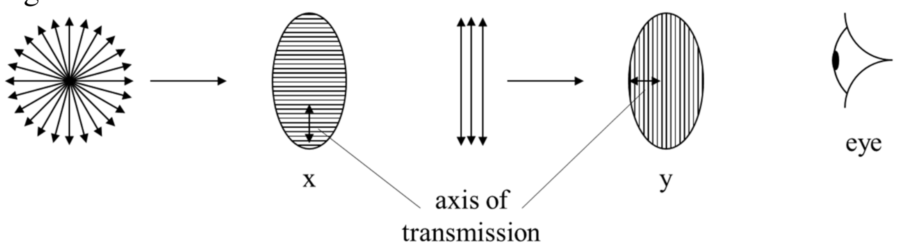
Note that only transverse waves can be polarised. Longitudinal waves, e.g. sound waves, cannot be polarised.

## Polarisation using Filters

We can produce a **linearly polarised** wave if we can somehow absorb the vibrations in all the other directions except one.

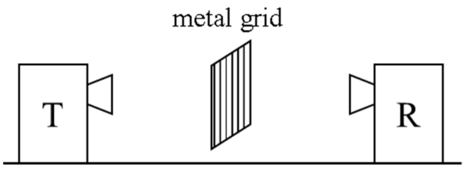
In 1852 William Bird Herapath discovered that a crystal of iodo-quinine sulphate transmitted one plane of polarisation, the other planes being absorbed. In 1938 Edwin Land produced the material 'Polaroid', which has a series of parallel long hydrocarbon chains. Iodine atoms impregnate the long chains providing conduction electrons. Light is only *transmitted* when the electric field strength vector is *perpendicular* to the chain.

The arrangement below shows a polaroid filter at X producing linearly polarised light. The polaroid at X is called a **polariser.** Vibrations of the electric field strength vector at right angles to the axis of transmission are absorbed.




A second polaroid at Y is placed perpendicular to the first one, as shown above. This is called an **analyser.** The analyser absorbs the remaining vibrations because its axis of transmission is at right angles to the polariser at X and no light is seen by the eye. The light between X and Y is called **linearly** or **plane polarisation**.

These effects also can be demonstrated using microwaves and a metal grid.



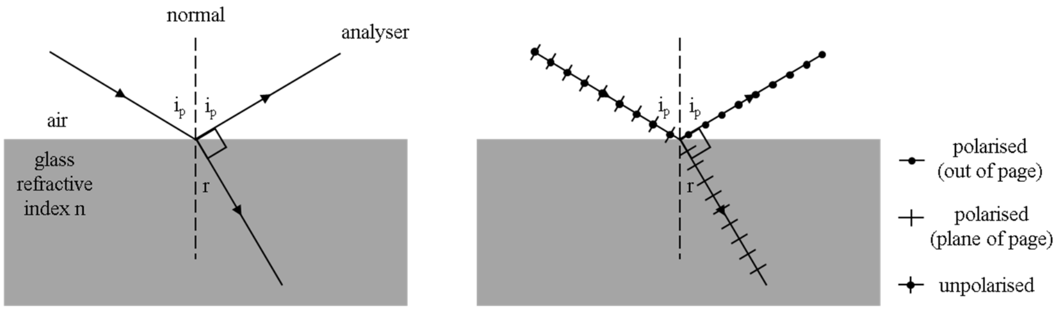

The microwaves emitted by the horn are plane polarised. In this example the electric field strength vector is in the *vertical plane*. The waves are **absorbed** by the rods and re-radiated in all directions. Hence there will be a very low reading on the receiver, R. When the metal grid is rotated through $90^o$ the waves will be transmitted, and the reading on the receiver will rise. Notice that the microwaves are *transmitted* when the plane of oscillation of the electric field strength vector is *perpendicular* to the direction of the rods.

Modern mobile devices and computer monitors produce polarised light. By placing a polarising filter in front of the screen you can observe variations in the transmitted and absorbed light at different angles. Some screens do not have all colours polarised in the same plane and the screen colour will change dynamically when you change the angle of the filter.

Polarisation by Reflection

Plane polarised waves can be produced naturally by light reflecting from any electrical insulator, like glass. When refraction takes place at a boundary between two transparent materials the components of the electric field strength vector parallel to the boundary are largely reflected. Thus reflected light is partially plane polarised.

Plane polarisation at the Brewster angle




Consider a beam of unpolarised light incident on a sheet of smooth glass. This beam is partially reflected and partially refracted. The angle of incidence is varied and the reflected ray viewed through an analyser, as shown above. It is observed that at a certain angle of incidence $i_p$ the reflected ray is plane polarised. No light emerges from the analyser at this angle.

The **polarising angle** $i_p$ or **Brewster's angle** is the angle of incidence which causes the reflected light to be linearly polarised.

This effect was first noted by an experimenter called Malus in the early part of the nineteenth century. Later Brewster discovered that **at** the polarising angle $i_p$ the refracted and reflected rays **are separated by 90°.**

Consider the diagram above, which has this 90° angle marked:

$$n = \frac{\sin i_p}{\sin r} \quad \text{but} \quad r = (90 - i_p) \quad \text{thus} \quad \sin r = \sin(90 - i_p) = \cos i_p$$

$$\text{thus} \quad n = \frac{\sin i_p}{\cos i_p} = \tan i_p \qquad \boxed{n = \tan i_p}$$

***Example***
*Calculate the polarising angle for glycerol, n = 1.47.*
*What is the angle of refraction inside the glycerol at the Brewster angle?*

**Solution**
Using the equation $n = \tan i_p$ $1.47 = \tan i_p$ giving $i_p = 56^o$.
At the Brewster angle, which is the polarising angle,
angle of refraction + $i_p$ = 90° thus angle of refraction = 34°.

## Reduction of Glare by Polaroid sunglasses

When sunlight is reflected from a horizontal surface, e.g. a smooth lake of water, into the eye, eyestrain can occur due to the glare associated with the reflected light. The intensity of this reflected beam can be reduced by wearing polaroid sunglasses. These act as an analyser and will cut out a large part of the reflected polarised light.