<u>Advanced Higher Physics</u>:

<u>MECHANICS</u>

<u>Simple Harmonic Motion</u>

At the end of this section, you should be able to:

Describe examples of simple harmonic motion (SHM).

State that in SHM the unbalanced force is proportional to the displacement of the object and acts in the opposite direction.

State and explain the equation $\mathbf{d^2y/dt^2 = -\ \omega^2\ y}$ for SHM.

Show that $\mathbf{y = A\ cos\ \omega t}$ and $\mathbf{y = A\ sin\ \omega t}$ are solutions of the equation for SHM.

Show that $\mathbf{v = \pm\ \omega\ \ /(A^2 - y^2)}$ for the above relationships.

Derive the expressions $\mathbf{1/2m\omega^2(A^2 - y^2)}$ and $\mathbf{1/2m\omega^2y^2}$ for the kinetic and potential energies for a particle executing SHM.

State that damping on an oscillatory system causes the amplitude of the oscillation to decay.

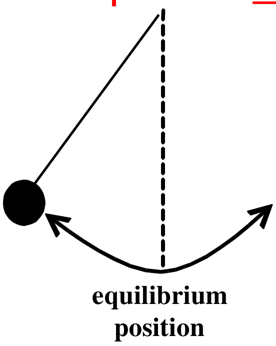
● **Describe examples of simple harmonic motion (SHM).**

If an object moves continuously back and forward the same distance either side of a fixed central point (equilibrium position) in the same time interval (period), the object is undergoing **simple harmonic motion** - **SHM** for short.

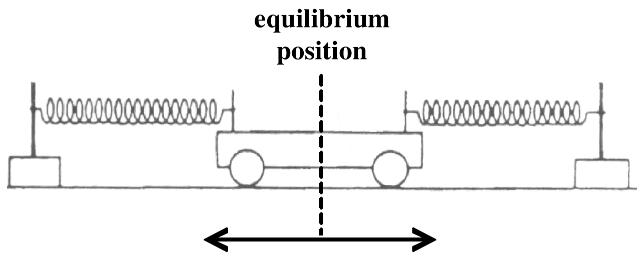
An **unbalanced force** acting on the object always causes the object to **accelerate towards** its **equilibrium position**.
(The **acceleration does not have** a **constant value**.)

Some examples of simple harmonic motion:

simple pendulum

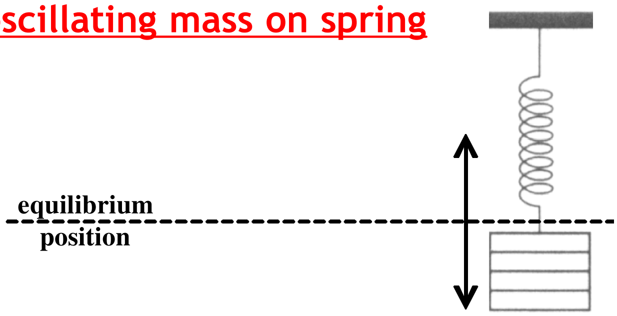


trolley between 2 springs

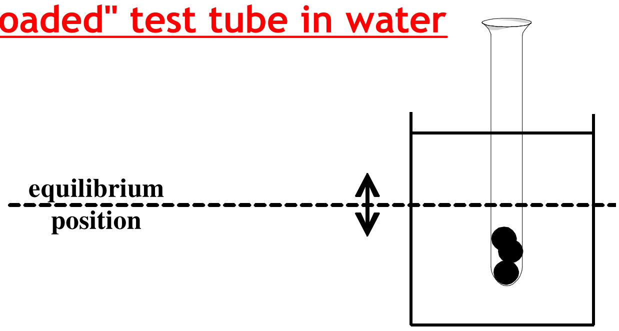


In each case, we assume that no frictional forces act - These would continuously reduce the distance moved by the object every "back and forward" or "up and down" cycle.
We consider this real life "damping" situation later on.

oscillating mass on spring



"loaded" test tube in water



● **State that in simple harmonic motion (SHM), the unbalanced force is proportional to the displacement of the object and acts in the opposite direction.**

For an object undergoing **simple harmonic motion**, the **unbalanced force** acting on the object is proportional to the **displacement** of the object from its **equilibrium position** and acts in the **opposite direction**.

In mathematical terms:

unbalanced force α displacement

$$\boxed{F = -k\,y}$$

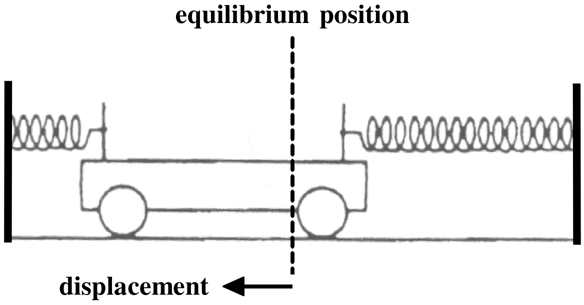
unbalanced force acting on object/ N

displacement of object from equilibrium position/ m

negative sign shows unbalanced force and displacement are in opposite direction

constant of proportionality/ $N\ m^{-1}$
(If system involves a spring, k is called the "spring constant")

1) At the instant shown, the trolley is moving to the left, away from its equilibrium position.



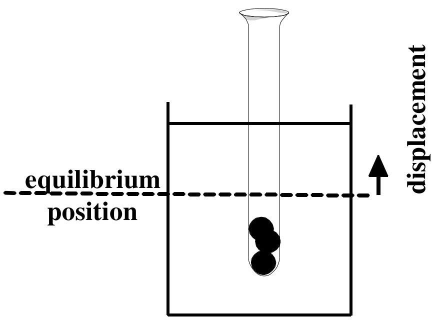
(a) Assuming the absence of all frictional forces, describe the subsequent motion of the trolley.

(b) What term do we use to describe such motion?

(c) For the instant shown on the diagram, describe the unbalanced force acting on the trolley in terms of the trolley's displacement from its equilibrium position.

(d) On the diagram, draw an arrow to show the direction of the unbalanced force acting on the trolley at the instant shown.

2) A "loaded" test tube floating in water undergoes simple harmonic motion in the vertical direction. At the instant shown, the test tube is moving upwards, away from its equilibrium position.

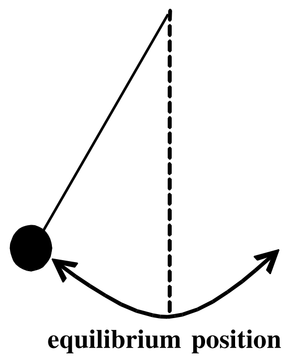


(a) Make a statement about the unbalanced force acting on the test tube.

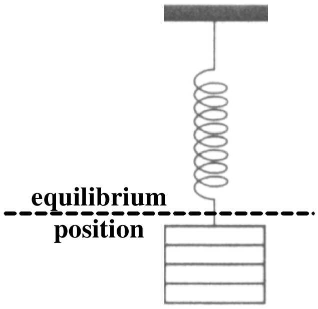
(b) Use an arrow to show the direction of the unbalanced force acting on the test tube at the instant shown.

(c) If, at the instant shown, the upward vertical displacement of the test tube was twice that shown, what affect would this have on the size and direction of the unbalanced force acting on the test tube?

3) Explain in terms of unbalanced force and displacement why a simple pendulum swings back and forward through its lowest point (equilibrium position).



4) With no mass hanging on it, the spring shown has a length of 12 cm. When the mass is hung on the spring, it exerts a force of 10 N on the spring which increases in length to 17 cm. If the mass is pulled downwards then released, the mass-spring system undergoes simple harmonic motion about its equilibrium position due to the action of an unbalanced force.



(a) On the diagram, draw an arrow to show the direction of the unbalanced force acting on the stretched spring immediately after the mass is released.

(b) State the relationship between the unbalanced force and the displacement of the mass-spring system from its equilibrium position.

(c) Calculate the extension of the spring when the mass exerting the 10 N force is hung on it.

(d) Using your answers to (b) and (c), determine the size of the unbalanced force immediately after release if the mass was pulled the following distances below its equilibrium position:
(i) 10 cm (ii) 2.5 cm (iii) 15 cm (iv) 1 cm.

HINT - Calculate the value for the constant of proportionality.

answers: (c) 0.05 m (d) k = 200 N m^{-1}, (i) 20 N (ii) 5 N (iii) 30 N (iv) 2N

5) To simulate simple harmonic motion on his "pogo stick", Ross jumps up and down continuously on the same spot. (A "pogo-stick" has a stiff spring at the bottom which compresses and stretches as someone jumps up and down on the stick).

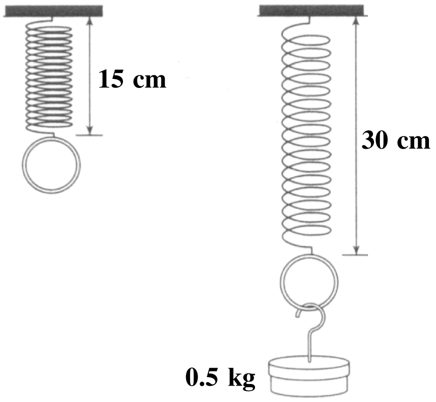
Before Ross mounts the "pogo stick", its spring has a length of 50.0 cm. When Ross mounts the "pogo stick", his weight of 700.0 N causes the spring to compress to 30.0 cm.

Determine the size of the unbalanced force acting on the "pogo stick" spring during simple harmonic motion when it compresses the following distances from its equilibrium position:

(a) 5 cm (b) 8 cm (c) 10 cm (d) 12 cm (e) 15 cm

answers: k = 3 500 N m^{-1} (a) 175 N (b) 280 N (c) 350 N (d) 420 N (e) 525 N

6) When suspended from the end of a clamp stand, a metal spring is 15 cm long. When Zubayr hangs a 0.5 kg mass on the spring, the spring increases in length to 30 cm.



(a) (i) Calculate the force the 0.5 kg mass exerts on the spring.

(ii) For the spring: **size of stretching force = constant of proportionality x displacement**. Determine the numerical value and unit of the **"constant"** in this relationship.

(b) Zubayr now pulls the 0.5 kg mass vertically down by 2.0 cm and releases it, causing the mass-spring system to undergo simple harmonic motion.

(i) Determine the size of the unbalanced force acting on the mass immediately after Zubayr releases it.

(ii) Hence, determine the size of the acceleration of the mass at this instant.

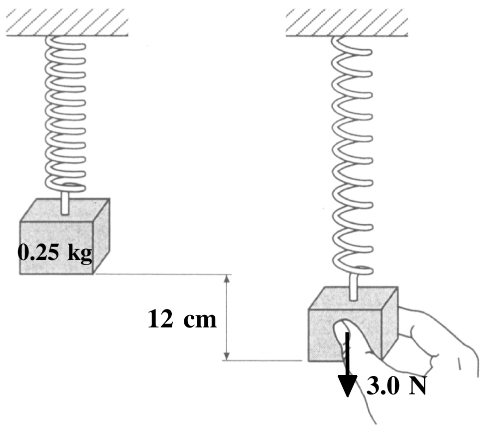
answers: (a) (i) 4.9 N (ii) 33 N m^{-1} (b) (i) 0.66 N (ii) 1.3 m s^{-2}

7) A wooden block of mass 0.25 kg is hung from a spring. When Euan pulls the block down with a force of 3.0 N, the spring stretches by 12 cm. When Euan lets go of the block, it performs simple harmonic motion on the spring.

(a) Calculate the force Euan must apply to pull the wooden block down by the distances stated below.

(b) For each distance, determine the size of the acceleration of the wooden block immediately after Euan releases it.

Distances are: (i) 3 cm (ii) 6 cm (iii) 8 cm (iv) 16 cm.



answers: k = 25 N m^{-1} (a) (i) 0.75 N (ii) 1.5 N (iii) 2 N (iv) 4 N (b) (i) 3 m s^{-2} (ii) 6 m s^{-2} (iii) 8 m s^{-2} (iv) 16 m s^{-2}

- **State and explain this equation for simple harmonic motion: $d^2y/dt^2 = -\omega^2 y$.**

unbalanced force α acceleration

hence:

For an object undergoing **simple harmonic motion**, the **unbalanced force** acting on the object (and hence its **acceleration**) is proportional to the **displacement** of the object from its **equilibrium position** and acts in the **opposite direction**.

This gives us the equation which defines **simple harmonic motion**:

acceleration α displacement

$$\boxed{\frac{d^2y}{dt^2} = -\omega^2 y}$$

acceleration of object/ ms^{-2}

displacement of object from equilibrium position/ m

negative sign shows acceleration and displacement are in opposite direction

constant of proportionality/ rad s^{-1}
(ω is called the "angular frequency")

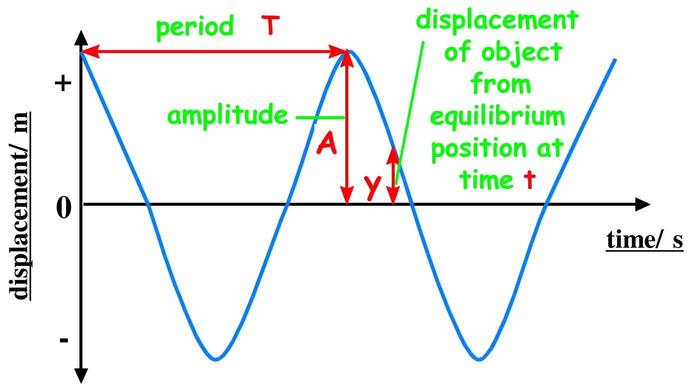
ω (the "angular frequency" constant) is not equal to the frequency (f) of an object undergoing simple harmonic motion.

ω is related to frequency (f) and period (T) as follows:

$$\omega = 2\pi f = 2\pi/T$$

The **displacement-time graph** for an object undergoing **simple harmonic motion** takes the form of a **cosine function** or **sine function**:

cosine function



$$\boxed{y = A \cos \omega t}$$

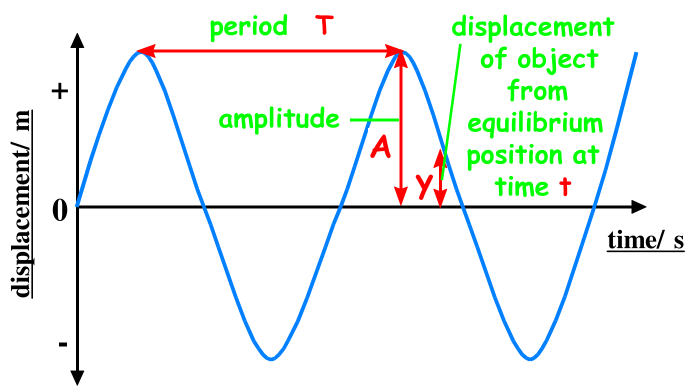
time/ s

angular frequency/ rad s^{-1}

displacement of object from equilibrium position at time t/ m

amplitude/ m

sine function



$$\boxed{y = A \sin \omega t}$$

time/ s

angular frequency/ rad s^{-1}

displacement of object from equilibrium position at time t/ m

amplitude/ m

- **Show that y = A cos ωt and y = A sin ωt are solutions of the equation for simple harmonic motion.**

To show that **y = A cos ωt** and **y = A sin ωt** are solutions of the equation for **simple harmonic motion**, we **differentiate them twice with respect to time** to obtain expressions for **acceleration**:

$y = A\cos\omega t$

$\therefore$ velocity $= \dfrac{dy}{dt} = -\,\omega A\sin\omega t$

$\therefore$ acceleration $= \dfrac{d^2y}{dt^2} = -\,\omega^2 A\cos\omega t$

$= -\omega^2 y$

(since $y = A\cos\omega t$)

$y = A\sin\omega t$

$\therefore$ velocity $= \dfrac{dy}{dt} = \omega A\cos\omega t$

$\therefore$ acceleration $= \dfrac{d^2y}{dt^2} = -\,\omega^2 A\sin\omega t$

$= -\omega^2 y$

(since $y = A\sin\omega t$)

Both **y = A cos ωt** and **y = A sin ωt** represent **simple harmonic motion** because, for each:

$$\frac{d^2y}{dt^2} = -\,\omega^2 y$$ (which is the equation for **simple harmonic motion**).

8) The motion of an object can be described by the equation: $\dfrac{d^2y}{dt^2} = -\,\omega^2 y$	
(a) What term describes the object's motion?	(b) How is the displacement of the object from its equilibrium position related to the unbalanced force acting on it at any instant?

9) Write down the equation which defines simple harmonic motion and explain each term.

10) The **displacement-time graphs** for four different objects undergoing simple harmonic motion are shown. In each case:

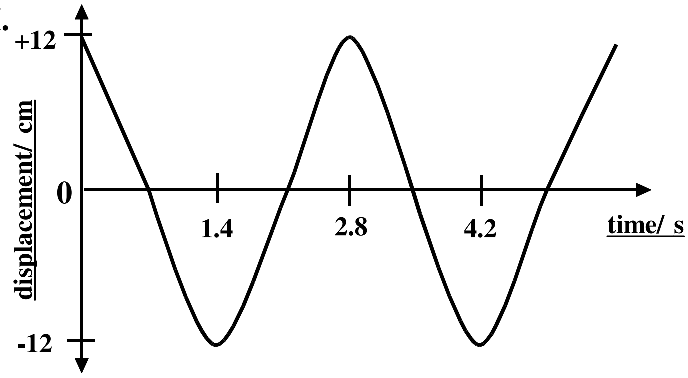
(a) Determine the value for the amplitude (**A**), period (**T**), frequency (**f**) and angular frequency (**ω**) of the motion.

(b) Use values from part (a) to obtain an expression in the form **$y = A \cos \omega t$** for the displacement **y** from the equilibrium position of the object undergoing simple harmonic motion.

(c) (i) Using your equation from part (b), derive an expression which gives the relationship between the **acceleration** of the object and **time t**.

(ii) Calculate a value for the **maximum acceleration** of the object.

I.



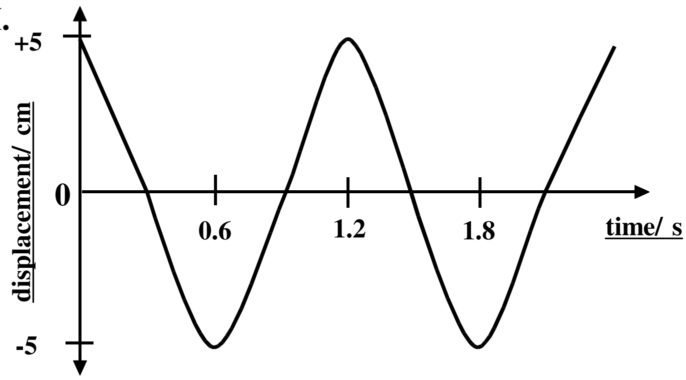
(a)

(b)

(c) (i)

(c) (ii)

II.



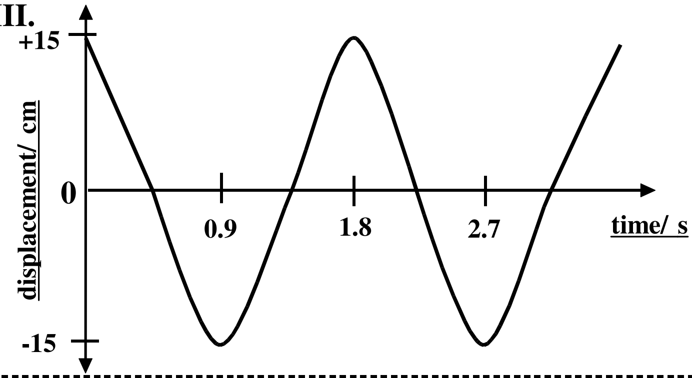
(a)

(b)

(c) (i)

(c) (ii)

III.



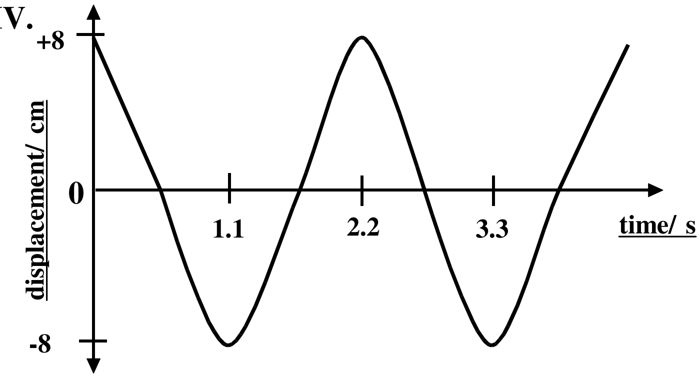
(a)

(b)

(c) (i)

(c) (ii)

IV.



(a)

(b)

(c) (i)

(c) (ii)

<u>Answers</u>	<u>Answers</u>	<u>Answers</u>	<u>Answers</u>
I. (a) A = 0.12 m	II. (a) A = 0.05 m	III. (a) A = 0.15 m	IV. (a) A = 0.08 m
T = 2.8 s	T = 1.2 s	T = 1.8 s	T = 2.2 s
f = 0.36 Hz	f = 0.83 Hz	f = 0.56 Hz	f = 0.45 Hz
ω = 2.2 rad s^{-1}	ω = 5.2 rad s^{-1}	ω = 3.5 rad s^{-1}	ω = 2.9 rad s^{-1}
(b) y = 0.12 cos 2.2t	(b) y = 0.05 cos 5.2t	(b) y = 0.15 cos 3.5t	(b) y = 0.08 cos 2.9t
(c) (i) - 0.58 cos 2.2t	(c) (i) - 1.4 cos 5.2t	(c) (i) - 1.8 cos 3.5t	(c) (i) - 0.67 cos 2.9t
(ii) - 0.58 m s^{-2}	(ii) - 1.4 m s^{-2}	(ii) - 1.8 m s^{-2}	(ii) - 0.67 m s^{-2}

11) A 2.5 kg weight is hanging on a spring at rest. Claire pulls the weight 15 cm downwards, then releases it, causing it to undergo simple harmonic motion. Immediately after Claire releases it, the weight accelerates upwards due to an unbalanced force of 12 N acting on it.

(a) Calculate the value for the upward acceleration of the weight immediately after Claire releases it.	(b) Determine the value of the angular frequency (**ω**) for the simple harmonic motion.	(c) Determine the frequency (**f**) and period (**T**) of the simple harmonic motion.

answers: (a) 4.8 m s^{-2} (b) 5.7 rad s^{-1} (c) f = 0.91 Hz, T = 1.1 s

12) Graham hangs a 1.2 kg steel mass on a spring. The mass and spring are at rest. Graham then pulls the mass down by 25 cm and releases it. Immediately after release, the mass is acted upon by an unbalanced force of 6.0 N upwards, causing it to accelerate towards its equilibrium position as it undergoes simple harmonic motion.

Determine the period and frequency of the simple harmonic motion.

answers: T = 1.4 s, f = 0.72 Hz

13) An object of mass 0.3 kg performs simple harmonic motion with a period of 1.7 seconds.

(a) Determine the size of the object's acceleration at the instant when its displacement is 16 cm from its equilibrium position.	(b) Calculate the size of the unbalanced force acting on the object at this instant.

answers: (a) 2.2 m s^{-2} (b) 0.66 N

14) An object of mass 0.25 kg performs simple harmonic motion with a frequency of 0.35 Hz.

(a) Determine the size of the object's acceleration at the instant when its displacement is 12 cm from its equilibrium position.

(b) Calculate the size of the unbalanced force acting on the object at this instant.

answers: (a) 1.5 m s^{-2} (b) 0.38 N

15) The period of a 0.15 kg object undergoing simple harmonic motion is 1.4 seconds. Calculate the size of the unbalanced force acting on the object at the instant when its displacement from its equilibrium position is 18 cm.

answer: 0.56 N

16) Show that $y = A \cos \omega t$ is a solution for the equation of simple harmonic motion.

17) Show that $y = A \sin \omega t$ is a solution for the equation of simple harmonic motion.

● **Show that $v = \pm \omega \sqrt{(A^2 - y^2)}$**

Starting from **$y = A \sin \omega t$** , we can derive an equation which relates the **velocity** (**v**) of an object undergoing **simple harmonic motion** to its **displacement from its equilibrium position** (**y**) at any time.

$y = A \sin \omega t$

$\therefore$ velocity (v) $= \dfrac{dy}{dt} = \omega A \cos \omega t$

Rearranging: $\sin \omega t = y/A$ and $\cos \omega t = v/\omega A$

$\cos^2 \theta + \sin^2 \theta = 1$

$\therefore \cos^2 (\omega t) + \sin^2 (\omega t) = 1$

$\therefore v^2/\omega^2 A^2 + y^2/A^2 = 1$ multiply through by $\omega^2 A^2$

$\therefore v^2 + y^2\omega^2 = \omega^2 A^2$

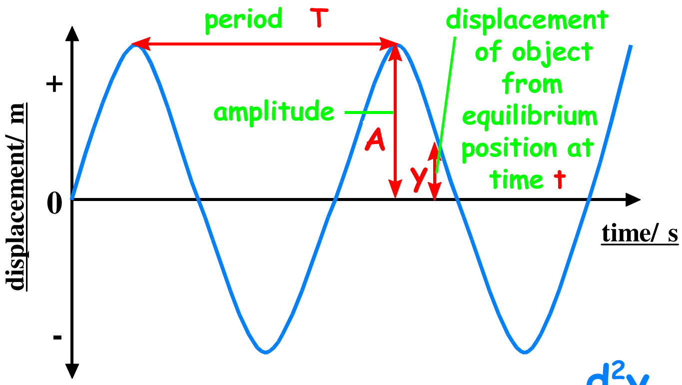
$\therefore v^2 = \omega^2 A^2 - y^2\omega^2$

$\therefore v^2 = \omega^2 (A^2 - y^2)$

$\therefore v = \pm \omega \sqrt{(A^2 - y^2)}$

The $\pm$ shows that for any displacement from the equilibrium position other than **$y = A$**, the velocity (**v**) could be in **2 opposite directions** since the square root of **$(A^2 - y^2)$** has a **positive** and a **negative** value.

The maximum velocity occurs when the object is **moving through its equilibrium position**. At this instant, **displacement** (**y**) **= o**.



$$\frac{d^2y}{dt^2} = -\omega^2 y$$

18) An object is undergoing simple harmonic motion with a frequency of 0.2 Hz. If the amplitude of its oscillations is 36 cm, calculate the object's velocity at the instant it is displaced 12 cm from its equilibrium position.

Answer: $\omega = 1.3 \text{ rad s}^{-1}$, so $v = \underline{+}0.44 \text{ m s}^{-1}$

19) An object is undergoing simple harmonic motion with a period of 2.2 seconds. If the amplitude of its oscillations is 85 cm, calculate its velocity at the instant it is displaced 15 cm from its equilibrium position.

Answer: $\omega = 2.9 \text{ rad s}^{-1}$, so $v = \underline{+}2.4 \text{ m s}^{-1}$

20) The amplitude of an mass on a spring undergoing simple harmonic motion with a frequency of 0.8 Hz is 25 cm. Determine the velocity of the object at the instant when it is displaced 18 cm from its equilibrium position.

Answer: $\omega = 5.0 \text{ rad s}^{-1}$, so $v = \underline{+}0.85 \text{ m s}^{-1}$

21) A weight suspended from a spring undergoes simple harmonic motion with a period of 3.5 seconds. The oscillations have an amplitude of 650 mm. If the weight is displaced 120 mm from its equilibrium position, calculate its velocity at this instant.

Answer: $\omega = 1.8 \text{ rad s}^{-1}$, so $v = \underline{+}1.2 \text{ m s}^{-1}$

22) The displacement of an object undergoing simple harmonic motion is 4 cm from its equilibrium position. The amplitude of oscillation is 12 cm and the frequency is 0.6 Hz. Calculate the speed of the object at this instant.

Answer: $\omega = 3.8 \text{ rad s}^{-1}$, so $v = \underline{+}0.46 \text{ m s}^{-1}$

23) A machine component undergoing simple harmonic motion with a period of 1.8 seconds is 450 mm from its equilibrium position. If its motion has an amplitude of 750 mm, determine its speed at this instant.

Answer: $\omega = 3.5 \text{ rad s}^{-1}$, so $v = \underline{+}2.1 \text{ m s}^{-1}$

24) (a) For an object undergoing simple harmonic motion, state in terms of its displacement from its equilibrium position, when the velocity reached is a maximum.

(b) In each case below, the values stated refer to an object undergoing simple harmonic motion. Determine the **maximum** **velocity** of the object in each case:

frequency = 2.0 Hz, amplitude = 15 cm **Answer:** v = 1.9 m s^{-1}	period = 1.5 s, amplitude = 10 cm **Answer:** v = 0.42 m s^{-1}
frequency = 1.5 Hz, amplitude = 25 cm **Answer:** v = 2.4 m s^{-1}	period = 0.4 s, amplitude = 36 cm **Answer:** v = 5.7 m s^{-1}
frequency = 5.2 Hz, amplitude = 18 cm **Answer:** v = 5.9 m s^{-1}	period = 0.2 s, amplitude = 22 cm **Answer:** v = 6.9 m s^{-1}

25) A boat is tied up in a harbour. Due to the motion of water in the harbour, the boat undergoes simple harmonic motion in the vertical direction with a frequency of 0.15 Hz. The amplitude of oscillation is 0.75 m.

(a) Calculate the vertical velocity of the boat at the instant when its displacement is 0.25 m from its equilibrium position.

Answer: $\omega = 0.94$ rad s^{-1}, so $v = \pm 0.67$ m s^{-1}

(b) Calculate the maximum vertical velocity of the boat.

Answer: $v = 0.71$ m s^{-1}

26) Derive an equation for the velocity of an object undergoing simple harmonic motion.

- **Derive the expressions $1/2m\omega^2(A^2 - y^2)$ and $1/2m\omega^2y^2$ for the kinetic and potential energies for a particle executing <u>simple</u> <u>harmonic</u> <u>motion</u>.**

kinetic energy

For an object undergoing simple harmonic motion:

velocity (v) = $\pm \omega \sqrt{(A^2 - y^2)}$

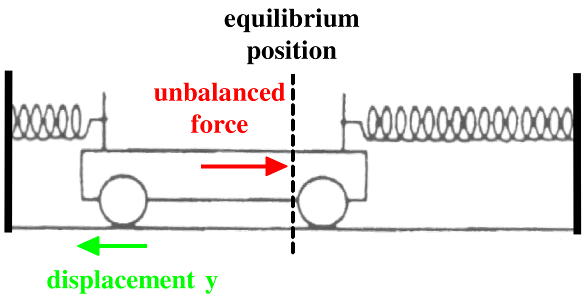
$\therefore v^2 = \omega^2 (A^2 - y^2)$

kinetic energy (E_k) = $1/2\ m\ v^2$

$\therefore \underline{E_k = 1/2\ m\ \omega^2 (A^2 - y^2)}$

potential energy

Consider this system undergoing simple harmonic motion:



unbalanced force α displacement

When trolley is at its equilibrium position:

displacement= 0 m $\quad \therefore$ unbalanced force = 0 N

At displacement y from equilibrium position:

unbalanced force = ma = m ω^2y ..... since $|a| = \omega^2 y$

$$\therefore \text{Average unbalanced force} = \frac{0 + m\,\omega^2 y}{2}$$

$$= 1/2\ m\ \omega^2 y$$

Work done on system = average unbalanced force x displacement

$$= 1/2\ m\ \omega^2 y \times y$$

$$= 1/2\ m\ \omega^2 y^2$$

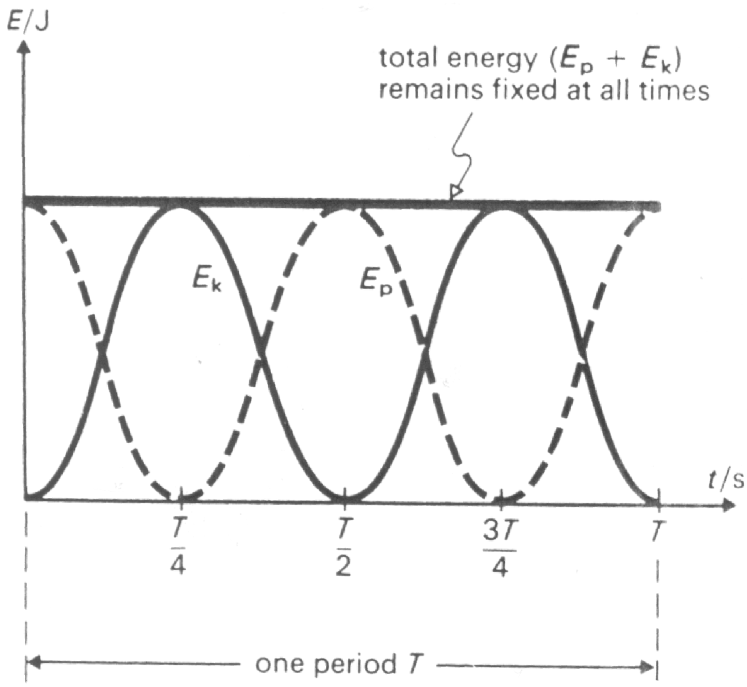
Work done during displacement y is stored in the spring system as potential energy (E_p)

$\therefore \underline{\text{potential energy} = 1/2\ m\ \omega^2 y^2}$

For an object undergoing simple harmonic motion, **assuming no energy is lost due to friction,** the **total energy** of the system remains **constant** at all times - but the **energy** is continually changing between **kinetic** and **potential**.

As the object passes through its **equilibrium position**, its **velocity** is a **maximum** - so its **kinetic energy** has its **maximum value** and its **potential energy** is **zero**.

At the instant the object reaches its **maximum displacement from its equilibrium position**, its **velocity** is **zero** - so its **kinetic energy** is **zero** and its **potential energy** has its **maximum value**.



Between these extremes, the **energy** of the object is a **mixture of kinetic and potential**.

27) An object of mass 0.25 kg is undergoing simple harmonic motion with a frequency of 1.2 Hz. The amplitude of its oscillation is 0.30 m.

(a) Calculate the kinetic energy of the object at the instant it is displaced 0.15 m from its equilibrium position.

(b) Calculate the potential energy of the object at this same instant.

(c) Calculate the total energy of the oscillating system.

Answers: (a) ω = 7.5 rad s^{-1}, so E_k = 0.47 J (b) E_p = 0.16 J (c) 0.63 J

28) A 1.2 kg object undergoes simple harmonic motion with a period of 2.5 seconds and an amplitude of 0.45 m.

(a) Calculate the kinetic energy of the object at the instant it is displaced 0.30 m from its equilibrium position.

(b) Calculate the potential energy of the object at this same instant.

(c) Calculate the total energy of the oscillating system.

Answers: (a) $\omega = 2.5$ rad s^{-1}, so $E_k = 0.42$ J (b) $E_p = 0.34$ J (c) 0.76 J

29) (a) Calculate the potential energy of a 0.75 kg object undergoing simple harmonic motion with a frequency of 2.4 Hz and an amplitude of 15 cm when it is displaced 12 cm from its equilibrium position.

(b) Calculate the kinetic energy of the object at this same instant.

(c) Calculate the total energy of the system.

Answers: (a) $\omega = 15$ rad s^{-1}, so $E_p = 1.2$ J (b) $E_k = 0.68$ J (c) 1.9 J

30) (a) Calculate the potential energy of a 1.3 kg object undergoing simple harmonic motion with a period of 0.35 seconds and an amplitude of 75 cm when it is displaced 55 cm from its equilibrium position.

(b) Calculate the kinetic energy of the object at this same instant.

(c) Calculate the total energy of the system.

Answers: (a) ω = 18 rad s^{-1}, so E_p = 64 J (b) E_k = 55 J (c) 119 J

31) A 0.36 kg object undergoes simple harmonic motion.

For this motion, amplitude = 15 cm and frequency = 12 Hz.

(a) (i) When is the velocity of the object a maximum?

(ii) Determine the kinetic energy and potential energy of the object at this instant.

(b) (i) When is the velocity of the object zero?

(ii) Determine the kinetic energy and potential energy of the object at this instant.

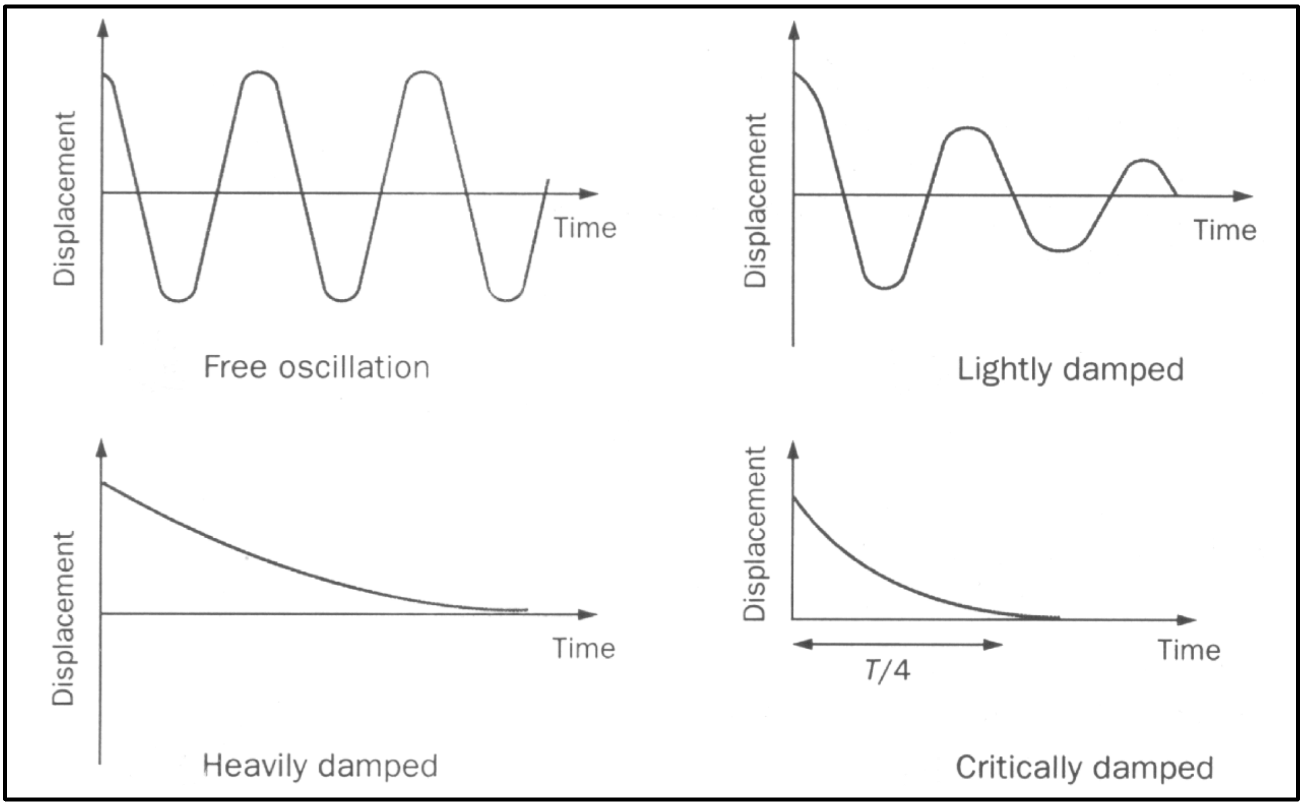
Answers: ω = 75 rad s^{-1} (a) (ii) E_k = 23 J, E_p = 0 J (b) (ii) E_k = 0 J, E_p = 23 J

32) Derive the expressions $1/2m\omega^2(A^2 - y^2)$ and $1/2m\omega^2y^2$ for the kinetic and potential energies for a particle executing simple harmonic motion.

- **State that damping on an oscillatory system causes the amplitude of the oscillation to decay.**

A system undergoing **simple harmonic motion** cannot do so for ever. **Opposing forces** (such as **friction** and **air resistance**) cause the **amplitude of the oscillation** to **decay** as time progresses - This is known as **damping**.

Damping converts the **total energy** of the system to **heat energy**.



Free Oscillation - There is no damping. The amplitude of the oscillations remains constant.

Light Damping - The amplitude of the oscillations decreases gradually with time to zero.

Heavy Damping - No oscillations occur. When the system is displaced from its equilibrium position and released, it returns very slowly to the equilibrium position, then remains there.

Critical Damping - No oscillations occur. When the system is displaced from its equilibrium position and released, it returns to the equilibrium position in as short a time as possible, then remains there
- The time is **T/4** (**1/4 of the period for free oscillation**).

The **suspension/shock absorber** system of vehicles makes use of **critical damping**. If the car travels over a bump in the road, the system must absorb the energy from the bump as quickly as possible without making the vehicle oscillate - So the ride is smooth.