INTRODUCTION

The scope of this publication

This publication has been written for students and teachers involved in the Higher Physics and Advanced Higher Physics courses. It is divided into three parts:

Part One (Sections 1–9) begins with a brief résumé of scientific notation and significant figures. This is followed by an explanation of where uncertainties in experimental results come from. Then there is description of how to estimate the uncertainty in a measurement and how to combine uncertainties in different measurements.

Part Two (Sections 10–15) deals with using graphs and spreadsheets to evaluate constants and quantify their uncertainties. Then follows a section on comparing the results of different experiments. There are numerical examples throughout.

Part Three (Sections 16–21) explains briefly the theory of uncertainties and the various procedures described in Part One. It shows how to deal with the uncertainties in functions and includes a short section on radioactivity measurements. It concludes by discussing the use of calculators for uncertainty calculations.

PART ONE

1. Accuracy of measurements and uncertainties

Physicists are unable to measure quantities exactly and so it is important to state explicitly the accuracy of a measurement.

The numerical result of any experiment should always be quoted as

value ± uncertainty

Experimenters can never determine the *true value* of a quantity because the accuracy with which a quantity can be established will always be limited by experimental error or uncertainty. However, the *value* quoted should be the best estimate of the measurement or outcome of the experiment. The *uncertainty* gives a measure of how certain the experimenter is that the true value is given by a value close to this best estimate.

If the uncertainty is large, the true value may lie within only a large range of the best estimate but if the uncertainty is small the experimenter is claiming that the true value is close to the best estimate. The numerical result may be based on a single measurement, *e.g.* a measurement of the length of an object using a ruler. In this case, the smallest scale division marked on the ruler will give an indication of the uncertainty. Alternatively, the result may be obtained after combining several measurements of this type. One then must know how to combine the individual uncertainties. In other experiments, the *value* and the *uncertainty* may require much analysis and the application of advanced statistical theory. In this case, the result will have a statistical interpretation and the range will indicate the probability of the true result lying within the limits given.

2. Significant figures and scientific notation

Significant figures

When calculating the result of an experiment or when quoting the result of an experiment, one should always consider to how many digits the result can justifiably be quoted.

Numbers should be rounded to be compatible with the uncertainty in the value, *i.e.* the number of significant figures quoted should match the degree of certainty of the measurement or result. If numbers are expressed in 'scientific notation', the digit before the decimal point is the most significant digit and the last digit after the decimal point is the least significant digit. This digit may be zero. More generally, the most significant digit is the leftmost, non-zero digit in any number. If the number does not have a decimal point, and does not end in zero, the last digit is the least significant one. If the number has no decimal point and ends in one or more zeros then ambiguity could arise. The last zero may or may not be the least significant digit. (See below.) If the number does have a decimal point, the least significant is the last digit, even if this is zero. The number of significant figures is obtained by counting the digits between the least and the most significant figure, including both of these. For example, each of the following numbers has four significant figures:

4.321×10^2
4.210
0.04321
43.21
43,210

Ambiguities over trailing zeros (zeros at the right of a number not followed by any non-zero digit) can be eliminated by using **scientific notation.** Express the number as one digit before a decimal point, followed by all necessary digits including the first uncertain one, followed by ten to the relevant power. (Better style would suggest that this rule be avoided where the power of ten is one of the following: –1, 0, +1.)

Note that a corollary of the above is that the last digit quoted in a number should be interpreted as being uncertain, *e.g.* in the number 2.15×10^3 the final 5 could be 4 or 6.

Determining **which digit is the first uncertain digit** is the substance of what follows in this text.

For a fuller discussion of significant figures see references 11 and 12 (on page 46).

3. Uncertainties

In the content statements of the Higher Physics and the Advanced Higher Physics and in this document, the use of the word 'error' is largely avoided although it is used extensively in the literature. The word 'error' in this context is not synonymous with *mistake* or *blunder*; the word, where used, is in a scientific context and gives an estimate of the accuracy of the measurement or result of experiment. *Uncertainty* conveys this more clearly and avoids possible misunderstanding; it is used here except when referring to *systematic error and error bars*. Of course, mistakes and blunders are made by scientists but their effects cannot be quantified unless they are detected. Often, they can only be detected by repeating the experiment and finding that one result is anomalous, *e.g.* one point on a straight line graph may be out of line, or by comparison of the result of one experimenter with that of another.

Reading uncertainty: The analogue scales on measuring instruments can only be read to some fraction of the smallest scale division. Often this is taken to be ½ of the smallest division but it may be possible in some cases to estimate a smaller fraction reliably. The reading uncertainty for instruments with a digital scale would normally be taken as ±1 of the smallest change in reading.

Calibration uncertainty: The manufacturers of scientific instruments calibrate the instrument against approved standards and give an indication of the accuracy of this calibration. They give the accuracy of calibration in terms of a range within which any one of the instruments is expected to lie. For example, the maker of a steel ruler may state that the length of 1 metre on the scale of the ruler is accurate to ± 0.5 mm. Most of the instruments sold by the manufacturer will be correctly calibrated or be very nearly correctly calibrated and large deviations from the correct calibration will be unlikely, although it is possible that with age and use an instrument may deviate significantly from its original calibration.

Random uncertainty: If an experiment is repeated many times, the result may not be the same each time. The experimenter may not set up the apparatus in exactly the same way each time, may start or stop a stopwatch with small differences of delay, *etc.* and these random differences give a range of results. If the effects are truly random, the set of results can be analysed statistically and the best estimate and the uncertainty estimated.

Systematic effects: These are different from the others in that they affect all of the results of the experiment in the same direction, *i.e.* the

results are all too small or too large. A simple example would be a measurement of length using a ruler. If the experimenter assumes that the zero of the scale coincides with the end of the ruler and it does not, then all the measurements will be too small. Alternatively, if the end of the ruler has been chopped off so that 5 mm is missing then all measurements will be too large. Of course, the experimenter should have noticed that a mistake is being made and corrected for this error. However, not all systematic effects are as simple as this and it is usually not possible to detect or evaluate them. If a systematic effect exists in only one of the variables being measured, then it may be possible to detect it using appropriate graphical treatment of the results – see page 29.

4. Scientific notation, units and multiples of units

In formal scientific reports, numerical results are only given in terms of the SI units and a number of agreed additional units – metres, kilograms and seconds being the most familiar. These units are listed in Appendix 1. It is convenient also to use multiples of these units and it is conventional to restrict these to multiples of 10^3 and denote these by a prefix to the unit. Strictly, this allows the metre, symbol m, the millimetre, mm, and the micrometre, *etc.* This does not allow for the centimetre, cm. It is likely, however, that the centimetre will continue to be used in all but the most formal of reports.

5. How uncertainties arise in experiments

Let us begin by looking at one particular experiment in some detail. We can suppose that Anne, a sixth-year student, is measuring the acceleration due to gravity, g, using a simple pendulum. She hangs the spherical metal bob by a cotton thread from a stand and measures the length of this pendulum (from the point of support to the centre of the bob) with a wooden metre stick. Then she sets the pendulum swinging through a small angle and times ten oscillations with a digital stopwatch. She records her results:

length l	= 51.25 cm
time for ten oscillations	= 14.78 seconds
hence period t	= 1.478 s.

then, from the equation for the period of a simple pendulum,

$$T = 2\pi\sqrt{l/g}$$

she calculates her value of g to be 9.262 m s^{-2} (to 4 significant figures).

Now Anne cannot measure the length, l, with perfect accuracy. Firstly, she reckons that she can read the scale of the metre stick to the nearest half-division, that is to 0.5 mm, as each division is 1 mm. This is the **reading uncertainty** in the measurement. Secondly, when Anne checks her metre stick against a steel scale (more accurate and more expensive), she finds that it is about 0.5 mm too long, 1.0000 m on her ruler corresponding to 0.9995 m on the steel scale.

This is a **calibration uncertainty** that will affect all measurements made with the metre stick. If it is identified, as in this example, by comparing the ruler with a steel rule, the measurements can be corrected for this **error** or deviation from the correct calibration.

It is reasonable to assume that her metre stick is uniformly too large and that her readings of length should be multiplied by 0.9995 to compensate. However, if the error is not identified, the error would still affect the measurements – all measurements would be too large in this case – and would be an example of a systematic effect. If the maximum value of the calibration uncertainty is quoted by the manufacturers, this value can be used as one of the uncertainties in the experiment.

There are uncertainties too in the time measurement. The smallest time interval that the digital stopwatch can display is 0.01 seconds. This is its reading uncertainty. (Anne cannot estimate to a fraction of a 'time division' as she can with the length division on the metre stick.) Again, when Anne checks her watch against the GMT time signals, she finds that it is running about 1.5 seconds slow in 24 hours. This is its calibration uncertainty; it is much less than the reading uncertainty, amounting to only about 0.0002 seconds during Anne's timing measurement and is therefore insignificant. This is another example of a systematic effect that has been identified. The measurements could be adjusted accordingly. Of course, if the GMT time signals are wrong (unlikely), her results will be subject to the unknown systematic error that this introduces.

As neither of these uncertainties seems to be very large, Anne is left wondering why her value for g is so much less than the value she expects, about 9.8 m s^{-2}. She therefore takes two further pairs of

measurements, which we list together with her original pair:

length/cm	*time for 10 oscillations/seconds*
51.25	14.78 (original)
51.40	14.37
51.30	14.52

Let us look at the measurements of t first. They are spread over a range of about 0.4 seconds, which is much more than the reading uncertainties and the calibration error of the stopwatch. So there must be other uncertainties, coming not from the watch itself but from the way that Anne measures the times. She has, for instance, to judge when the pendulum is at the centre of its swing, and then to start and stop the watch at the right moments. She can do none of these things precisely; she does them slightly differently each time. These and other random effects mean that her timings are all slightly different. We say that there is an **uncertainty** caused by these random effects in her measurement of time which, in this case, is larger than the reading uncertainty associated with a single reading of the stopwatch.

When Anne measures the length, l, she finds random effects there too. The metre stick is old and its ends are worn, so that the position of its zero mark is slightly uncertain. In addition, she has to estimate by eye where the centre of the spherical bob is in relation to the metre scale. Hence her measurement of length is also subject to uncertainty.

Now Anne can reduce these uncertainties by improving the experiment. If she uses a light beam and a photoelectric cell to detect when the pendulum passes through its centre of oscillation, she can time the swings more accurately. She can use different lengths of thread and, by appropriate data handling, get round the problem of locating the centre of the bob. But she can never completely eliminate uncertainty. Vibrations and other disturbances, together with variations in temperature, pressure, humidity and so forth, will always prevent her obtaining exactly repeatable values of l and t. Hence although there is some 'true value' of g for Anne's experiment, she can never find it precisely!

6. A closer look at uncertainty

Let us suppose now that Anne has set up a light beam and photoelectric cell so that she can time the oscillations more accurately. She then obtains the following six values of g:

9.643 9.752 9.981 9.808 9.785 9.732 m s^{-2}.

The spread of values here is due to the many random effects remaining in Anne's experiment that are still beyond her control. What can she do with these six different figures to improve the accuracy of her result? First, she takes the mean value of her results to be the best value of g that she can obtain. (This may seem obvious but does in fact require proof.)

mean value = (9.643 + 9.752 + 9.981 + 9.808 + 9.785 + 9.732)/6
= 58.701/6 = 9.784 m s^{-2}

She now wants to represent the uncertainty in this mean value by a single figure. The simplest way to do this is to divide the **range** covered by her six values (largest minus smallest) by six, the number of values that she has measured.

This method works well for up to about 12 values (see, for example, reference 1 for more on this).

Thus: uncertainty $\Delta g = (9.981 - 9.643)/6 = 0.056$ m s^{-2}.

The mean value and the uncertainty can be written more compactly as:

$$g = 9.784 \pm 0.056 \text{ m s}^{-2}.$$

How many **significant figures** should be used here? The uncertainty of 0.056 m s^{-2} tells us that the true value of g probably lies somewhere between 9.728 and 9.840 m s^{-2}. 'Probably' here means a probability of around 70% (see Section 18 for details). Hence there is a corresponding probability of about 30% that the true value lies outside these limits. Moreover, further sets of measurements will yield different mean values and different uncertainties.

Because the range only gives a statistical probability, the number of significant figures justified in the uncertainty is limited; uncertainties

should usually be quoted to only one significant figure. However, in order to avoid rounding uncertainties, it is best to do all calculations to two significant figures until the final result is obtained. Then the final uncertainty can be rounded off to one significant figure. (However, if that figure is a '1', then the second figure should be retained). The mean value of the results should then be rounded off to an equivalent number of decimal places. Thus Anne would present the above result as:

$$g = 9.78 \pm 0.06 \text{ m s}^{-2}.$$

We now summarise the results of this section in algebraic form. For n values of a quantity x,

$$x_1, x_2, \ldots, x_i, \ldots, x_n,$$

the mean value $\bar{x}$ is given by

$$\bar{x} = (x_1 + x_2 + \ldots + x_i + \ldots + x_n)/n = \frac{\Sigma x_i}{n} \qquad (6.1)$$

the range is $(x_{max} - x_{min})$ and the uncertainty Δx is thus

$$\Delta x = (x_{max} - x_{min})/\,n \qquad (6.2)$$

7. Combining uncertainties

We have now identified three different types of uncertainty:

Calibration uncertainty – indicates how well an instrument has been made; see Appendix 2 for examples. If the instrument can be compared with a more accurate measuring device, we can compensate and remove this calibration uncertainty from the results. If not, this uncertainty remains as an unknown systematic error.

Reading uncertainty – indicates how well an instrument scale can be read.

Random uncertainty – indicates the combined effect of all the random factors affecting the experiment.

Ideally the experiment should be designed so that possible calibration uncertainties are identified and so that both calibration and reading uncertainties are much smaller than the random uncertainty. In practice, this cannot always be done. Hence we need to know how to obtain from these three uncertainties a figure that represents the total uncertainty in a measurement. The rule here is (see Section 8): The square of the total uncertainty equals the sum of the squares of the individual uncertainties.

Thus three uncertainties Δx, Δy and Δz produce a total uncertainty Δw given by

$$\Delta w^2 = \Delta x^2 + \Delta y^2 + \Delta z^2. \qquad (7.1)$$

We use this expression when the three uncertainties are of comparable size. If one uncertainty is about three or more times larger than the others, then this uncertainty will dominate and it is not necessary to carry out the calculation; we can take the total uncertainty to be the largest uncertainty alone.

Thus, for example, suppose that $\Delta x = 1$, $\Delta y = 3$ and $\Delta z = 1$. Then, using equation 7.1, $\Delta w = \sqrt{11} = 3.32$. Rounding this to one significant figure (Section 3) gives $\Delta w = 3$. Thus, to within the accuracy to which uncertainties can be estimated, we have $\Delta w = \Delta y$.

This is a very important simplification. Whenever uncertainties are combined by an equation such as 7.1, any uncertainty less than about one-third of the largest may be ignored. This often makes uncertainty calculations much easier.

Example 7a

We measure the width w of a metal bar with a steel ruler and obtain the result

$$w = 12.5 \text{ mm}.$$

Calibration uncertainty = 0.1 mm (see Appendix 2)
Reading uncertainty = 0.5 mm (our estimate)
Random uncertainty = unknown, since it cannot be estimated from one reading.

As the reading uncertainty is five times larger than the calibration uncertainty, we take it alone as the total uncertainty, and write the result as

$$w = 12.5 \pm 0.5 \text{ mm}$$

Example 7b

An analogue voltmeter, set to the 30 volt range, reads 17.5 volts. The scale is marked in half-volt divisions. We estimate that we can read this scale to one-fifth of a division, that is, to 0.1 volts.

Calibration uncertainty = 2% of 30 volts (Appendix 2)
= 0.6 volts
Reading uncertainty = 0.1 volts
Random uncertainty = unknown

Here the calibration uncertainty dominates and is taken to be the total uncertainty; the result is:

$$\text{voltage} = 17.5 \pm 0.6 \text{ volts}$$

Example 7c

The temperature of the beaker of water is measured with a mercury-in-glass thermometer, marked in divisions of 1°C. The reading is 18.5°C and the reading uncertainty is estimated as half a division.

Calibration uncertainty = 0.5°C (Appendix 2)
Reading uncertainty = 0.5°C
Random uncertainty = unknown

As the calibration and reading uncertainties are of comparable size, we apply equation 7.1 and obtain for the total uncertainty:

$$\sqrt{(0.5)^2 + (0.5)^2} = 0.707$$

We round this to one significant figure and give the result as:

$$\text{Temperature} = 18.5 \pm 0.7°\text{C}$$

These three examples show how to estimate the uncertainties in single readings. However, it is always advisable to repeat measurements in order to:

(i) detect blunders,
(ii) check that the quantity being measured is not changing,
(iii) estimate the uncertainty in the quantity being measured by analysing the spread and distribution of the results obtained.

Example 7d

We measure the width of the metal bar (see example 7a) at six different points down the length of the bar, again with the steel ruler. We obtain:

12.0, 12.5, 12.0, 12.0, 12.0, 12.0 mm

Calibration error = 0.1 mm (Appendix 2)
Reading uncertainty = 0.5 mm (our estimate)

Here, the repeated value of 12.0 indicates that the uncertainty is so small that it is totally masked by the reading uncertainty. The only other value obtained differs by the reading uncertainty and so we conclude that the bar has parallel sides and take the best estimate of the width to be the mean of the readings. We therefore take the reading uncertainty to be the total uncertainty and give the result as:

$$w = 12.1 \pm 0.5 \text{ mm}$$

If we want more accurate values of w and Δw, we will not get them by more arithmetic, nor by further measurements of the same kind. Rather, we should reduce the reading uncertainty, either by reading the scale to less than 0.5 mm or by using a more accurate instrument, as in the next example.

Example 7e

The width of the bar is now measured with vernier callipers, on which the scale can be read to the nearest 0.2 mm. Six values are obtained at different points along the bar:

12.30, 12.22, 12.42, 12.56, 12.68, 12.54 mm.

Calibration uncertainty = 0.01 mm (Appendix 2)
Reading uncertainty = 0.02 mm
Random uncertainty = 0.077 mm (using equation 6.2)

We take the random uncertainty alone as it is more than three times larger than either the calibration or the reading uncertainty. We express it to one significant figure only, and the mean value (using equation 6.1) to the same number of decimal places. We write the result as:

$$w = 12.45 \pm 0.08 \text{ mm}$$

We should note that this result does not enable us to distinguish between random effects in the measurements and non-uniformity in the bar. If we feel that we are skilful enough to use the callipers to about the same accuracy as the reading uncertainties, then we can attribute the uncertainty to non-uniformity. If we are not so confident, we must make repeated measurements at one point on the bar to discover how skilful we really are.

8. Uncertainty calculation

In most experiments, we enter our measured values into an equation in order to calculate the final result. To find the uncertainty in this result, we have to combine the uncertainties in these values in some way. This usually requires us to be able to find:

(i) the uncertainty in a quantity raised to some power,
(ii) the uncertainty in the product or quotient of two quantities,
(iii) the uncertainty in the sum or difference of two quantities.

Let us first note that we can write the uncertainty in a quantity x in several ways, as the **absolute uncertainty** Δx, or as the **fractional uncertainty** $\Delta x/x$, or expressed as a percentage, $(\Delta x/x)\times 100$ %.

8(i). The uncertainty in a quantity raised to a power

If the quantity x is raised to a power n, then the fractional uncertainty in x^n is n times the fractional uncertainty in x, *i.e.* n $(\Delta x/x)$.

Example 8a

Suppose $x = 2.50 \pm 0.12$. Find the uncertainties in

$$y = x^3 \text{ and } z = \frac{1}{\sqrt{x}}$$

We first find the fractional uncertainty in x;

$$\frac{\Delta x}{x} = \frac{0.12.}{2.50} = 0.048 \text{ or } 4.8\%$$

(It is often helpful to handle fractional uncertainties as percentages.)
We then have:

for $y = x^3$, $n = 3$ for $z = \frac{1}{\sqrt{x}}$, $n = -\frac{1}{2}$

$\frac{\Delta y}{y} = 4.8\% \times 3 = 14.4\%$ $\frac{\Delta z}{z} = 4.8\% \times ½ = 2.4\%$

but $y = 2.503 = 15.6$ but $z = \frac{1}{\sqrt{2.5}} = 0.781$

hence Δy = 14.4% of 15.6 = 2.24 hence Δz = 2.4% of 0.781 = 0.0187

Only when the calculation is finished do we round off to an appropriate number of significant figures (Section 2). We then write:

$y = 16 \pm 2$ $z = 0.781 \pm 0.019$

For negative n, we ignore the negative sign, as uncertainties are always expressed as plus-or-minus some number. Thus the fractional uncertainty in $1/x$ ($= x^{-1}$) equals the fractional uncertainty in x, even though their absolute uncertainties are different.

8(ii). The uncertainty in a product or quotient

Suppose two variables x and y are related by either $w = xy$ or $w = x/y$
In each case the fractional uncertainty in z is given by

$$\frac{\Delta w}{w} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2} \qquad (8.1)$$

Example 8b

A voltage of 17.5 ± 0.6 volts (example 7b) is measured across a 560 Ω resistor marked with a gold band. The current through the resistor is thus 17.5/560 = 0.0313 amperes = 31.3 mA. Find the uncertainty in this value.

The fractional uncertainties are:

$\Delta V/V = 0.6/17.5 = 0.034$ or 3.4%
$\Delta R/R = 0.05$ or 5% (see Appendix 2)

The fractional uncertainty in the current is then

$$(\Delta I/I) = \sqrt{0.034^2 + 0.05^2} = 0.06$$

Then, as 0.06 of 31.3 is 1.9, we may write the value of the current as

$$I = 31.3 \pm 1.9 \text{ mA}$$

As the first figure in the uncertainty is '1' we give the uncertainty here to two significant figures and express the current to a corresponding number of decimal places. Notice that we first calculate the two fractional uncertainties separately so that we can compare their magnitudes before combining them. If one of them is three or more times larger than the other, then (as in Section 7) we take the larger one only to be the total uncertainty. Alternatively we often compare % uncertainties, in this case 3.4% for voltage and 5% for resistance, to see if any can be neglected.

This method can be extended to any number of quantities combined by multiplication or division. Thus if $w = (xy)/z$, then

$$\frac{\Delta w}{w} = \sqrt{\left(\frac{\Delta x}{x}\right)^2 + \left(\frac{\Delta y}{y}\right)^2 + \left(\frac{\Delta z}{z}\right)^2}$$

8(iii). The uncertainty in a sum or difference

If w is related to x and y by

$$w = x \pm y$$

then the absolute uncertainty in w is given by

$$\Delta w = \sqrt{\Delta x^2 + \Delta y^2} \qquad (8.2)$$

Example 8c

The temperature in a beaker of water rises from $T_1 = 18.5$°C to $T_2 = 22.0$°C. Find the uncertainty, ΔT, in the temperature rise $T_2 - T_1$.

If, as in example 7e, the temperatures are measured with a mercury-in-glass thermometer, then T_1 and T_2 each have a reading uncertainty of 0.5°C and a calibration uncertainty of 0.5°C. However, it is reasonable to suppose that for a small temperature rise, the calibration uncertainties will largely cancel each other out. Hence we are left with the reading uncertainties alone so that:

$T_1 = 18.5 \pm 0.5$°C; $T_2 = 22.0 \pm 0.5$°C
Hence $\Delta T^2 = (0.5)^2 + (0.5)^2 = 0.5$ and $\Delta T = 0.7$°C,
so that $T_2 - T_1 = 3.5 \pm 0.7$°C

Note here that the fractional uncertainty $\Delta T/T$ is made very large (20%) by the small temperature rise. This points to poor experimental procedure, which should be improved either to produce a larger temperature rise or to enable T_1 and T_2 to be measured more accurately.

Armed with the techniques of 8(i) – (iii), we can now find the uncertainties in more complex equations.

Example 8d

John is measuring the specific heat capacity c of a liquid. He immerses a heater of resistance R in a mass m of the liquid, and passes a current I through it for t seconds. The temperature of the liquid rises from T_1 to T_2. The energy output of the heater, $I^2 R\, t$, equals the energy gain of the liquid, $mc\,(T_2 - T_1)$. Hence

$$c = \frac{I^2 R t}{m(T_2 - T_1)} \qquad (8.3)$$

To find the uncertainty Δc, John works as follows:

(i) he finds the fractional uncertainty in I^2 (example 8a),
(ii) he finds the absolute uncertainty in $T_2 - T_1$ (example 8c), and converts it to a fractional uncertainty,
(iii) he compares these two fractional uncertainties with those in R, m and t, and discards all that are less than one-third of the largest,
(iv) he combines the remaining fractional uncertainties (example 8b) to give the fractional uncertainty in c,
(v) finally, he converts this to an absolute uncertainty and presents the result in the form $c \pm \Delta c$.

9. Graphical methods

In the last example (8d), John would be very unenterprising if he were simply to measure the rise in temperature for just one period of time. It would be far better to read the thermometer at regular time intervals and then plot a graph of temperature against time. This graph will be a straight line (if the liquid is well insulated). Its gradient will equal the average rate of temperature rise and will in effect be the average of several pairs of temperature-time measurements. Hence it can replace $(T_2 - T_1) / t$ in equation 8.3.

If John adopts this graphical method, he then faces the problem of finding the uncertainty in the gradient of a graph. We now see how to do this.

A straight-line graph has the general form

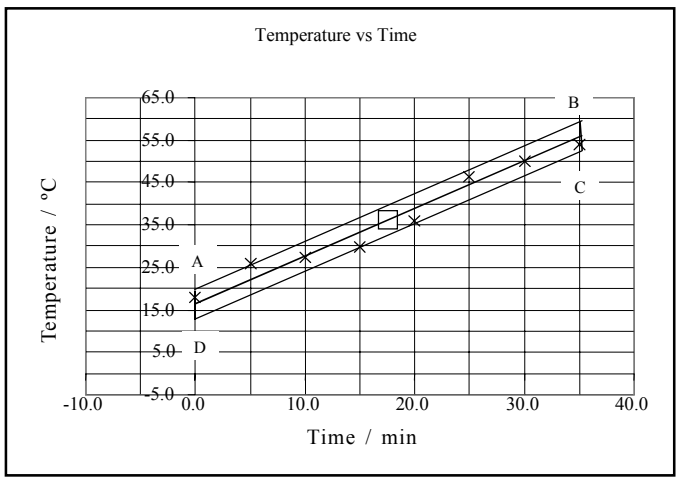
$$y = mx + b \text{ where } b \text{ is the intercept.}$$

In John's experiment, the water temperature T is the dependent variable represented by y and the heating time t is the independent variable represented by x. Suppose that he obtains the following results:

t /min	0	5	10	15	20	25	30	35
T / ^{0}C	18.0	26.0	27.5	30.0	36.0	46.5	50.0	54.0

We first plot the points in the usual way (Figure 1).

Figure 1



As the points are all of equal accuracy (reading uncertainties in t and T being the same for all readings) then it can be shown that the best straight line through them passes through their **centroid**. This is the 'centre of mass' of the points; its co-ordinates are the means of all the x and y values. Here the centroid is the point (17.5, 36.0) and is marked with a square.

We then draw what we judge to be the best straight line through the centroid. When this is done by eye, our judgement will almost certainly be based on minimising the deviations of the points from the line. As will be seen later, when the best straight line is calculated, it is the sum of the **squares** of the deviations that is minimised and not the magnitudes of the deviations. After drawing the line, the gradient and intercept, m and b, are found in the usual way:

$$m = 1.12°\text{C/min}; \; b = 16.3°\text{C}$$

We now enclose the points in a long thin box, a 'parallelogram of uncertainty' (reference 3). The ends of the box are vertical lines AD and BC through the first and last points of the graph. The top line AB is drawn parallel to the best line (this can be judged by eye – see later for a more reliable method using a spreadsheet program) so that it passes through the point that lies farthest above the best line – here it is the point (5, 26.0). The lower line DC is similarly drawn through the point that lies farthest below the best line – here it is the point (15, 30.0).

The 'worst' lines that could reasonably be drawn through the points are the diagonals of the parallelogram, AC and BD. From the gradients of these diagonals we can find the uncertainty Δm in the gradient of the best line. It is better not to draw in the diagonals, as the graph then becomes very cluttered. We just need the co-ordinates of the corners of the parallelogram. They are:

A,	(0, 20.0)	B,	(35, 60.0)
C,	(35, 52.0)	D,	(0, 13.0)

The gradients of the two diagonals are:

$$m(\text{AC}) = 0.91 \qquad m(\text{BD}) = 1.34.$$

Then, for n points on the graph, the uncertainty in m is given by:

$$\Delta m = \frac{m(BD) - m(AC)}{2\sqrt{(n-2)}} = \frac{0.43}{2\sqrt{6}} = 0.088$$

so that: $m = 1.12 \pm 0.09°\text{C/min}$

The fractional uncertainty $\Delta m/m$ is then 0.09/1.12 = 0.08, and the percentage uncertainty is 8%.

The factor 2 in the expression for Δm arises because the best line bisects (roughly) the angle between the two diagonals. The factor $\sqrt{(n-2)}$ comes from the statistical theory of the best line. We see that increasing the number of points reduces the uncertainty in the gradient, as we would expect.

If the uncertainty Δb in b is required, we can find it in a similar way. The diagonals AC and BD are extended back to the y-axis (not necessary in this example, as A and D are already on the y-axis) and the two intercepts found. Here they are b(A) = 20.0 and b(D) = 13.0. Then Δb is given by

$$\Delta b = \frac{b(A) - b(D)}{2\sqrt{(n-2)}} = \frac{7}{2\sqrt{6}}$$

Hence: b = 16.3 ± 1.6°C

If all we want is the **fractional** uncertainty in the gradient, $\Delta m/m$, then we can use a quicker method. Suppose we use the horizontal distance between the verticals AD and BC to find all three gradients m, m(BD) and m(AC). Then, from the geometry of Figure 1, we have:

$$\frac{\Delta m}{m} = \frac{m(BD) - m(AC)}{m} \times \frac{1}{2\sqrt{(n-2)}}$$

$$\frac{\Delta m}{m} = \frac{2 \times BC}{BA} \times \frac{1}{2\sqrt{(n-2)}}$$

$$\frac{\Delta m}{m} = \frac{y(B) - y(C)}{y(B) - y(A)} \times \frac{1}{2\sqrt{(n-2)}}$$

Using the co-ordinates of the corners, we have

$$\frac{\Delta m}{m} = \frac{60 - 52}{60 - 20.0} \times \frac{1}{\sqrt{6}} = \frac{8}{40.0} \times \frac{1}{\sqrt{6}} = 0.08$$

The fractional uncertainty in m is 8% as before.

Using this method, we do not even have to measure y(A), y(B) and y(C) in units of y; we can simply count millimetres up the graph paper.

Example 9a

John repeats his experiment (example 8d), using a graphical method to obtain the rate of temperature rise as a gradient, m degrees per second. Then the specific heat capacity of the liquid is

$$c = \frac{I^2 R}{Mm}$$

To find the uncertainty in c, John works as follows:

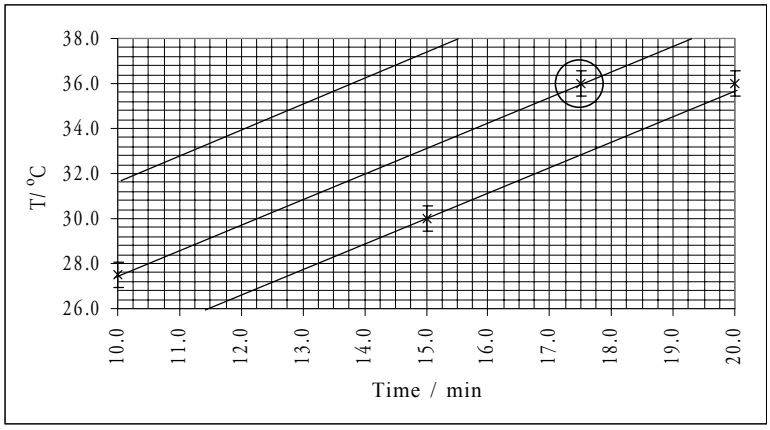
(i) he finds the fractional uncertainty in $I^{2,}$
(ii) he finds the fractional uncertainty in the gradient $\Delta m/m$,
(iii) he compares all the fractional uncertainties and discards as before,
(iv) he combines the remaining fractional uncertainties to obtain $\Delta c/c$,
(v) he calculates the absolute uncertainty Δc.

Suppose a single 'rogue' point lies a long way from the best line. Should it be included in the parallelogram or not? Such points are often produced by misplotting; hence this should be checked first. Then the actual values of x and y should be examined and any obvious blunder corrected. If the point survives these tests, it should still be discarded if it produces a parallelogram whose diagonals miss the centroid by a substantial amount.

It is often helpful to show the reading uncertainties in x and y by plotting error bars on the graph. They can often give a 'visual clue' as to how well the experiment has gone.

In John's experiment, the reading uncertainty in the time t is too small to show. The reading uncertainty in the temperature T is ± 0.5°C and is shown for three points in the enlarged central portion of the graph (Figure 2).

Figure 2



Here we see that the deviations of the points from the best line are much larger than the error bars. What if John really found such large deviations? He would suspect that something must be wrong with his experimental method to produce such large temperature fluctuations. Perhaps he sometimes forgot to stir the water, so that it was not always at a uniform temperature when he read the thermometer.

The error bars are not used in drawing the parallelogram of uncertainty. The actual uncertainties in the readings are represented by the scatter of the points around the best line. Hence the lines AB and CD should pass through the furthermost points themselves (see the lower line in Figure 2) and not through the tips of their error bars.

Similarly, no attempt should be made to combine any estimated uncertainty in t or T with Δm or Δb, which already express all the uncertainties in the points.

PART TWO

10. Use of graphs and spreadsheets

A graph is valuable as it provides a visual impression of the data.

- the existence or not of a trend is easily seen
- points which do not conform to the general trend are readily identified
- the equation to which a set of data conforms may be suggested by the distribution of the plotted points
- spreadsheet programs allow graphs to be drawn and data analysed with relative ease
- by including error-bars, an impression can be got of the relative sizes of random uncertainties, from the spread of the points, and of the other sources of uncertainty

It is always worth while drawing a graph of the results whilst the experiment is in progress. Any point not conforming to the trend can be identified and re-assessed, either by immediately re-doing the measurement, or if this is not possible due to the experimental conditions, then by re-doing that measurement before the equipment is dismantled. Care must be taken not to replace data unless one is certain that the previous data was in error – if not, one is in danger of simply confirming one's prior assumptions about the outcome of the experiment.

The HSDU publication *Physics Guide to Excel* (reference 4) gives an excellent introduction to setting up and using a spreadsheet to analyse experimental results.

Before using the spreadsheet to draw a graph, some thought should be given to the experiment being performed. The student should first of all assess whether the results are likely to fit a known relationship. Examples would be an experiment designed to investigate the pressure–volume relationship for a fixed mass of gas at constant temperature or one to investigate the inverse square law for gamma radiation. Frequently the results will be expected to conform to an equation, such as that between the period of a pendulum and the length of the pendulum. In this case constants will be involved, such as the acceleration due to gravity, g, and indeed it may be measurement of this quantity that is the desired outcome of the experiment. Since a straight-line graph is immediately recognisable, the data in these cases should be

handled in a way which would be expected to yield a straight line. Less commonly, the experiment may be designed to study some completely unknown (to the student at least) relationship, such as that between the viscosity of a liquid and the quantity of some additive. In these cases no constant is being measured, and any graph drawn will simply illustrate the trend.

11. Graphing results with the EXCEL spreadsheet – Adding a trendline and equation

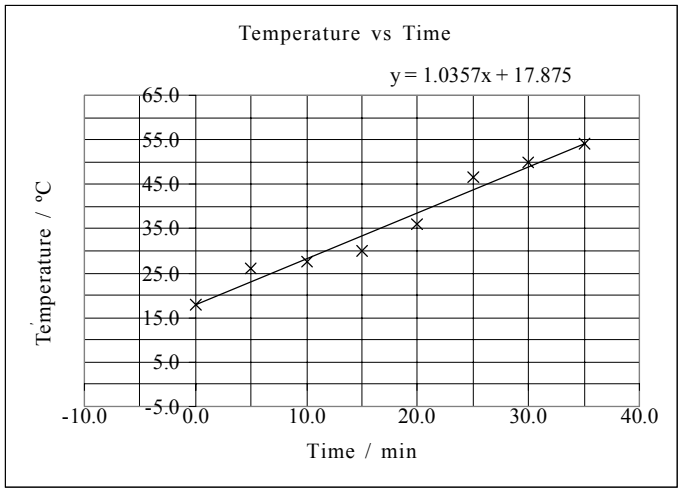
The HSDU publication *Physics Guide to Excel* (reference 4) provides a clear, step-by-step account of how to produce a chart of a set of experimental points using an Excel spreadsheet. That guide also explains how to add error bars to the plotted points. It does not explain how to add a trendline and equation to the computer generated chart.

After preparing the chart of the temperature readings versus time (see Figure 3), proceed as follows –

- select the chart area in the spreadsheet by clicking on it once,
- go to **Chart** on the Menu bar,
- click on **Add Trendline.**
- select **Type – Linear**
- now click on the **Options** tab and tick the box **Display equation on chart**

The least squares, best-fit line will be added to the set of plotted points. The equation of this line (given normally by $y = mx + c$ but in EXCEL by $y = mx + b$) is given also. By double clicking on the best-fit line, the characteristics of the line, colour and weight can be altered; if desired, the line can be constrained to pass through a known intercept, by clicking on **Options,** ticking the 'set intercept' box and specifying the intercept value in the '=' box.

Figure 3



The equation of the best-fit line gives

$$m = 1.04\ ^0\text{C/ min}; \quad b = 17.88\ ^0\text{C}$$

12. Uncertainties in gradient and intercept

Using a spreadsheet, the parallelogram (see Section 9, page 19) can be added using the DRAW package. A line is drawn lying over the trendline, extending between the first and last 'x' values. By copying this line (CTRL C followed by CTRL V) a second identical line is generated. These lines can then be moved until they pass through the most extreme upper and lower data points. Often these lines can be moved using the keyboard direction arrows. For finer steps CTRL + ARROW can be used. Finer grid-lines can be used to give the co-ordinates of the vertices of the parallelogram so formed.

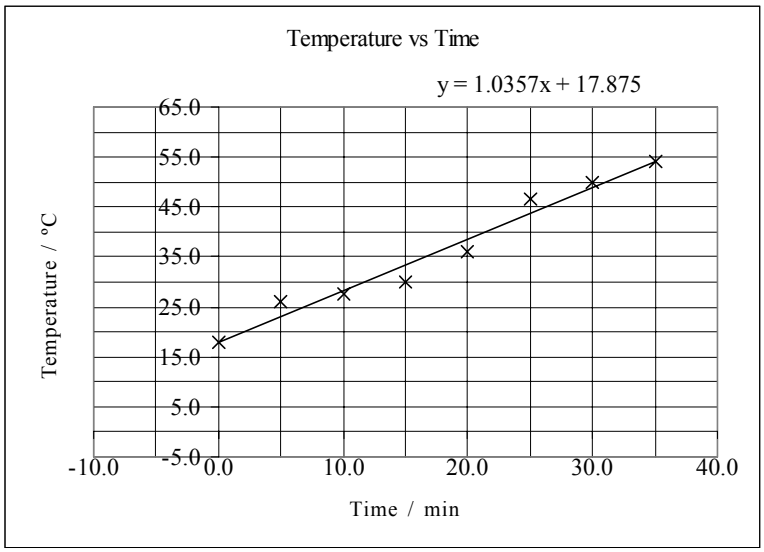
13. Note on LINEST

It is straightforward to have the analysis package of the spreadsheet find the values of the '**standard error' or 'standard deviation' of the gradient,** and the **'standard error' ('standard deviation')** of the **intercept**. (Note that in EXCEL the intercept is referred to by the symbol 'b' rather than 'c'). The function LINEST will supply the necessary statistics. The example here uses the temperature–time readings of the 'heating water' experiment:

t /min	0	5	10	15	20	25	30	35
T /^{0}C	18.0	26.0	27.5	30.0	36.0	46.5	50.0	54.0

The data are entered in the first two columns of an EXCEL spreadsheet as before, and the graph drawn, with a trendline inserted. The equation of the line can be shown on the chart by selecting – Chart Options – Insert Trendline, Options – tick 'Display equation on Chart' box.

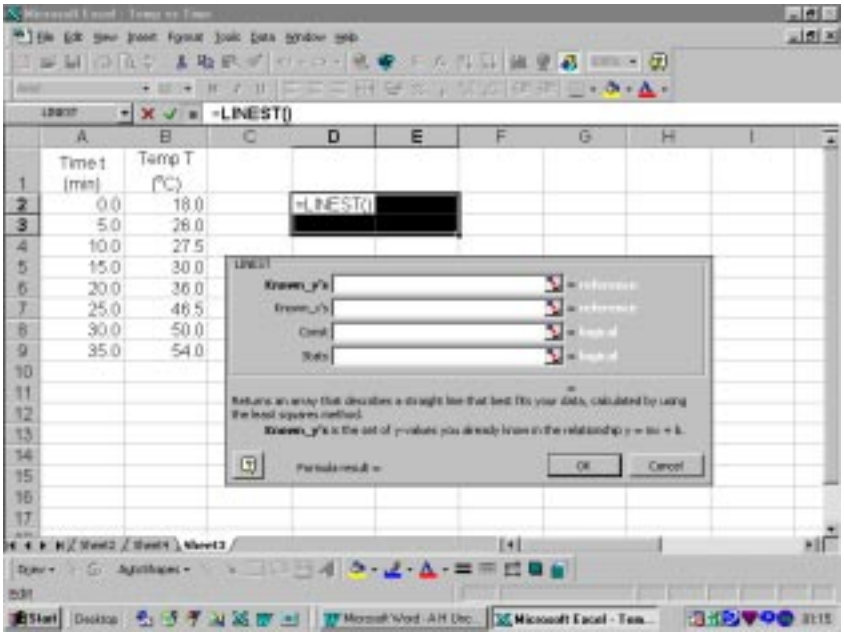
Figure 4



As above (Figure 4), this gives a gradient of 1.04, and an intercept *b* of 17.88.

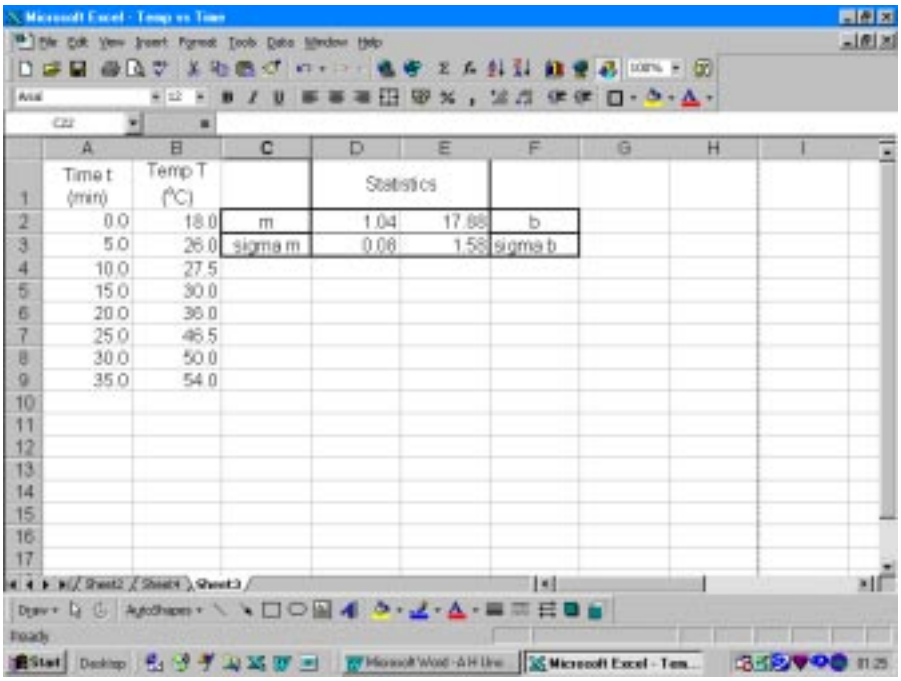
When the statistics of the line are found they will be fitted into an array. Highlight a block of cells two columns wide by 2 deep, to take this data. With these cells highlighted enter the formula =LINEST. The dialog box shown will open up:

Figure 5



Click on the red symbol to the right of the first box (**Known *y*'s)** and enter the range of cells containing the y-values. This is most easily done by running the cursor down column B from B2 to B9, or simply type these cell references by hand. Do the same for the known *x*'s. Enter 'TRUE' in each of the bottom two lines. **To enter this formula as an array, hold down CTRL + SHIFT and press ENTER at the same time.** The formula should appear in the formula bar with { } brackets around it, and the cells you highlighted should now hold the desired data.

Figure 6



The top left cell which you highlighted contains *m*, and the top right cell contains *b*. The standard deviation of each is the box immediately below it.

In the screen display above, the names etc have been added to the surrounding boxes. The program will not do this for you.

An alternative method is to highlight cells as described earlier, and then type =LINEST() in the formula line. Place the insertion point inside the brackets, then highlight the cells containing the *y*-data readings (in this case B2:B9), or simply enter these references, followed by a comma, then the range of the *x*-data (A2: A9), then a comma followed by TRUE, then another comma followed again by TRUE, giving =LINEST(B2:B9, A2:A9,TRUE,TRUE)

Now hold down CTRL + SHIFT and press ENTER. This should return the required values as before.

14. Data handling with a spreadsheet – seeking a straight line

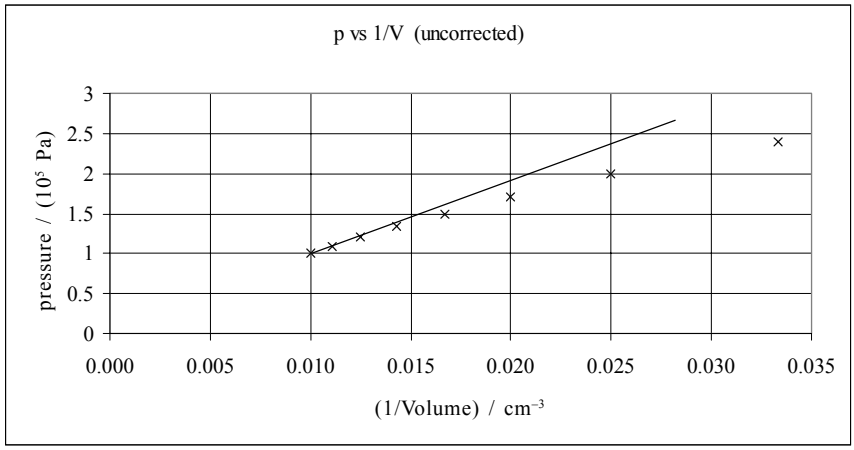
Should a straight line relationship be sought, the spreadsheet makes calculations related to this extremely quick and easy to carry out.

Let us suppose that an experiment investigating the relationship between the pressure and volume of fixed mass of gas yields the following values of p and V. Reasonable care has been taken over controlling the temperature and mass of the gas:

Pressure p / 10^5 Pa	1.00	1.09	1.20	1.33	1.50	1.71	2.00	2.40
Volume V / cm^3	100	90	80	70	60	50	40	30

The student plots p vs $1/V$, expecting a straight line:

Figure 7



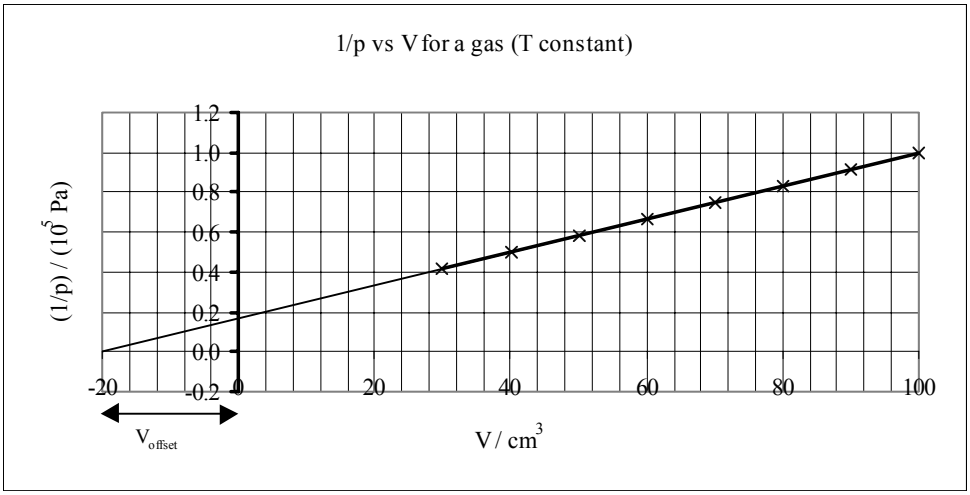
The line on the graph was drawn through the first four points.

The result is clearly not what was expected, and should give cause for some thought. The trend is getting further from a straight line for large $1/V$, *i.e.* for small V.

This could be because the assumed relationship pV = constant is incorrect, but alternatively there could be a systematic error in the results. For example, a 'dead-space' in the equipment could cause all the volume readings to be too small by the same amount.

A plot is made of $1/p$ vs V in order to examine this possibility:

Figure 8



This shows clearly the offset in the volume, which is identified with a dead-space in the equipment. In this case there was a dead-space of 20 cm^3. The apparatus may be redesigned to reduce or eliminate this. The results can be corrected to allow for the dead-space, and the linear relationship can be shown to hold:

Pressure p / 10^5 Pa	1.00	1.09	1.20	1.33	1.50	1.71	2.00	2.40
Volume V / cm^3	100	90	80	70	60	50	40	30
Corrected Volume / cm^3	120	110	100	90	80	70	60	50

The equation is now that of a straight line through the origin, with no offset.

The ease with which such graphs can be generated makes it possible to scrutinise data more closely when linearity is expected (and fails to show up). Examples where offsets are likely to occur are in the recording of gas volumes – 'dead-spaces' as exemplified above, distances between a light-source and a detector, distances between a source of radiation and a geiger-tube, and so on. In fact, rather than attempt to measure from a true zero, it may be more convenient and yield better data to deliberately introduce an offset. It is, for example, more straightforward, in a pendulum experiment, to mark the string above the bob and

measure from there. Length should then be plotted as the independent variable, and the square of the period should be plotted as the dependent variable. In this case, the offset could be found from the graph. If g is being measured, the offset is irrelevant, as g can be obtained from the gradient.

Note: This method will fail if there is an unknown systematic error in the measurement of the dependent variable as well.

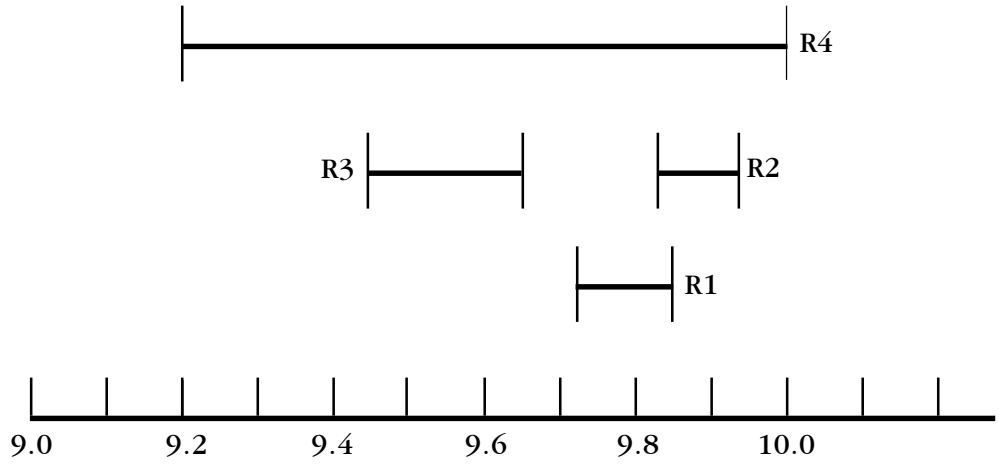
15. Comparisons of results

Once we have a numerical result in the form $x \pm \Delta x$, we can compare it with the results of other experiments. As an example, let us compare Anne's value for g with three other values, two obtained with a pendulum and one with a spiral spring. The four results are listed below and shown in Figure 9 with the uncertainties drawn as horizontal bands.

R1	Pendulum	9.78 ± 0.06 m s^{-2} (Anne's result)
R2	Pendulum	9.88 ± 0.05 m s^{-2}
R3	Pendulum	9.56 ± 0.07 m s^{-2}
R4	Spring	9.6 ± 0.4 m s^{-2}

Let us first look at these results on their own, disregarding whatever else we know about the value of g. Figure 9 shows that R1 and R2 have overlapping error bands. Hence we can say that R1 is in good agreement with R2. R3 lies somewhat lower and so is in rather poor agreement with R1 and R2.

Figure 9



The uncertainty band for R4 encompasses all the others. Hence R4 is in good agreement with R1 – R3, but its broad uncertainty band marks down the spiral spring as a poor way of measuring *g*. By comparing uncertainties in this way, we can judge between one experimental method and another.

The value of *g* used in SQA examination question papers is often given as 9.8 m s^{-2}. This is not the 'right answer' but rather a 'generally accepted value', derived from many careful experiments that have produced results with overlapping uncertainty bands that are very small. More comprehensive data books (for example, reference 7) reveal that very few of the so-called 'constants' are actually constant; most of them depend on other physical parameters. Thus *g* varies with both latitude and altitude (see reference 8). Across Scotland, at sea level, it ranges from 9.815 m s^{-2} in Stranraer to 9.819 m s^{-2} in Lerwick (with uncertainties less than 0.001 m s^{-2} in each value.)

As the uncertainty band for R1 includes these values, we say that R1 is in good agreement with the generally accepted values of *g* for Scotland. The same is true for R2, for the lower edge of its uncertainty band (9.83 m s^{-2}) is also very close to the accepted values.

There is however, a sizeable gap between the uncertainty band for R3 and the accepted values; the agreement is poor. Perhaps there was some undetected error in this experiment that displaced the result to a lower value. However, we cannot be sure and so cannot say that R3 is 'wrong'. Instead we note the discrepancy and, if we think it important enough, ask for the experiment to be repeated. The history of science shows that there is often much to learn from so-called 'wrong' answers.

PART THREE

16. Populations, samples and distributions

In Section 5, we looked at a simple experiment in physics, Anne's measurement of g with a pendulum. Let us compare this with an experiment in a rather different field. John, another student, is conducting a statistical survey; he is measuring the heights of a large number of S5 students. Do these two experiments, apparently unrelated, have anything in common?

One point of similarity is that both John and Anne are working with **samples**. John has chosen a sample from the total **population** of S5 students. Anne also has obtained a sample (though not by conscious choice) from the total 'population' of measurements that she might have made. John's population is finite; there are only around 40,000 S5 students in Scotland. Anne's population is infinite; in principle she could go on taking readings for ever.

John's sample displays a **distribution** of student heights. This **sample distribution** is related somehow to the **population distribution**, the distribution of height throughout the whole population of S5 students. Similarly Anne's sample shows a distribution of values of g and this too is related in some way to the distribution in the total population of results that she might have obtained. If John and Anne repeat their experiments, John with a fresh batch of students and Anne with a fresh set of measurements, then they are sampling their respective populations a second time. The second samples may not look much like the first one, but they are just as representative of the populations from which they were drawn.

However, the sample distributions in these two experiments have very different origins. 'Acceleration due to gravity' is a single-valued entity. Even though it is not a constant, it has only one value at any particular place and time. The distribution of values that Anne finds in her sample comes from the uncertainties inherent in her experimental method (Section 6). 'Height of S5 student', however, is not a single-valued entity. The distribution of heights in John's sample comes mainly from the distribution inherent in the S5 population and only peripherally from his experimental method.

If now John and Anne want to improve their experiments, they will proceed very differently. For Anne, 'improvement' means a closer approach to a single true value of g in her laboratory. She will redesign her apparatus and improve her measurement techniques. For John, 'improvement' means a better knowledge of the distribution he is studying. His simple apparatus for measuring heights is perfectly adequate and needs no improvement; instead he will concentrate on improving his sample techniques.

Yet, despite these differences, both John and Anne face a similar task. Both experiments yield a set of numbers, a sample distribution, which they must interpret. This requires some knowledge of the mathematical theory of distributions (see 'Further reading and references', pages 45–46). Only a brief outline of the theory is presented here.

17. The mathematics of distributions

A distribution can be described most simply by two numbers, one being the **mean value** of the distribution and the other representing the spread or **width** of the distribution. The first of these we have already met: for a sample of n values of a quantity x, the mean value m is given by:

$$m = \frac{\sum_{i=1,\ n} x_i}{n} \qquad (17.1)$$

The width of the distribution is computed from the deviations of the n values of x from the mean m. By far the most useful measure of width is the root mean square deviation, known universally as the **standard deviation** s;

$$s = \sqrt{\frac{\sum (x_i - m)^2}{n}} \qquad (17.2)$$

Now m and s relate to the **sample** distribution. **The population** distribution, from which the sample is drawn, also has a mean and a standard deviation, denoted by the Greek letters μ and σ respectively.

John could, in principle, measure the height of every S5 student in Scotland – the whole population – and so find exact values of μ and σ. But Anne's population is infinite; she can never find exactly μ and σ. Instead she can only estimate them. The best estimate of the population mean is the sample mean m;

$$\mu(\text{est}) = m \qquad (17.3)$$

The best estimate of σ *(est)* can be shown to be $s\sqrt{n/n-1}$, or

$$\sigma\ (est) = \sqrt{\frac{\sum (x_i - m)^2}{(n-1)}} \qquad (17.4)$$

Furthermore she can estimate how much the sample mean *m* varies about the population mean μ. This is called the **standard uncertainty of the mean**, and is given by $\sigma_m = \sigma(est)/\sqrt{n}$

$$\sigma_m = \sqrt{\frac{\sum (x_i - m)^2}{n(n-1)}} \qquad (17.5)$$

On calculators, the keys marked σ_n and σ_{n-1} compute *s* and σ (est) respectively. There is usually no key for σ_m.

All these quantities can be computed for any distribution such as John's or Anne's. But before we can say what they mean in terms of probability (as in Section 18), we must know, or rather make some assumption about, the basic shape of the population distribution.

18. The normal distribution

Given sufficient time and patience, John and Anne can both take very large samples from their respective populations, and can then plot out their sample distributions as histograms. These will show fairly well what the population distribution curves look like. They are usually bell-shaped and symmetrical, with a single central peak.

We can never predict the exact shape of the distribution curve for any particular experiment. There is however one curve, the **normal** or **Gaussian** distribution, that fits many sample distributions quite well. The theory of the normal distribution rests on the assumption that any measurement is subject to a multitude of tiny perturbing effects, each of which is equally likely to make the measured value slightly too large or too small. The cumulative effect of all these perturbations, acting independently and randomly, produces the different values that we obtain whenever we make many measurements of a physical quantity. (We are thinking here particularly of the random physical factors that

affect Anne's measurements. A similar argument can be applied to the random physiological factors that produce the distribution of student heights in John's experiment.)

Figure 10

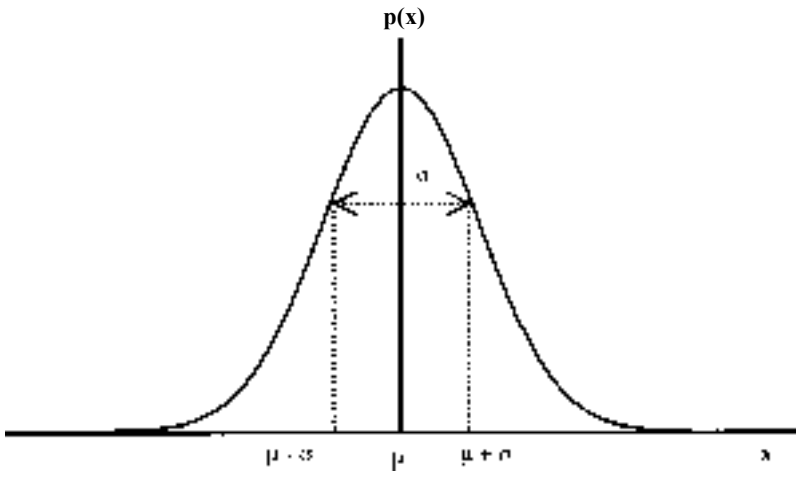


Figure 10 shows a graph of the normal distribution curve, in which p(x), the probability of obtaining a value x, is plotted against x. The curve peaks at $x = \mu$, the mean value, which is (in the absence of systematic effects) the true value of the quantity being measured. The equation of the curve is:

$$p(x) = \frac{1}{\sqrt{(2\pi\sigma^2)}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad (18.1)$$

The standard deviation of this distribution is found by an integration equivalent to the summation of equation 17.2. It turns out to be just the constant σ in equation 18.1. This is a measure of the width of the curve, which goes through inflexions at the points $\mu \pm \sigma$. If σ is small, the curve will be sharply peaked; as s increases, the peak falls and the curve becomes broad and shallow. About 68% of the area under the curve lies within the range $\mu \pm \sigma$.

If we now assume that, for a particular experiment, the population of measurements does indeed follow a normal distribution, we can now say more precisely what is meant by the parameters σ (est) and σ_m of Section 17.

(i) The best estimate of σ, which measures the width of the normal distribution, is σ(est). There is thus a 68% probability that any one measurement x_i will lie within the range μ ± σ (est). On average, 68% of the sample will lie within ± σ (est) of the true value of the quantity being measured.

(ii) There is also a 68% probability that the sample mean m will lie within the range $\mu \pm \sigma_m$. As σ_m is less than σ (est) by a factor $\sqrt{n}$, it follows that m is likely to be closer to μ than any one measure x_i. (Hence the value of repeated measurements.)

Let us consider how reliable these estimates of probability are likely to be. If we take several samples from the same population, they will all have different means m and different standard uncertainties σ_m. So if we have just one value of σ_m, how accurate is it? In other words, what is the 'uncertainty in the uncertainty'?

If can be shown that, for a normal distribution, the fractional uncertainty in σ_m is

$$\frac{\Delta\sigma_m}{\sigma_m} = \frac{1}{\sqrt{2(n-1)}} \qquad (18.2)$$

For six readings, this amounts to 32% and only falls to 10% for nearly 50 readings. Hence the standard uncertainty is usually only significant to one figure, and we can use approximate methods to estimate it. The simplest of these is the 'range method', already described in Section 6. For $n < 12$, this produces good, but slightly large estimates of σ_m, as the following example shows.

Example 18a

For Anne's six values of g, listed in Section 6, we obtain the following results:

Standard deviation of sample	s = 0.102 (from eqn 17.2)
Standard deviation of population	σ (est) = 0.112 (from eqn 17.4)
Standard uncertainty in mean	σ_m = 0.046 (from eqn 17.5)
Uncertainty (range method)	Δg = 0.056 (from eqn 6.2)

Using the value of the uncertainty given by the range method, Anne can give her measurement as:

$$g = 9.78 \pm 0.06 \text{ m s}^{-2}$$

where the uncertainty indicates that there is around a 70% probability that the true value of g lies in the range 9.72 – 9.84 m s^{-2}.

19. Uncertainties in combinations and functions of variables

Suppose the quantity w is a function of two variables x and y:

$$w = f(x,y)$$

If the measurements of x and y follow normal distributions with standard deviations σ_x and σ_y, then it can be shown that the standard deviation is given by:

$$\sigma_w^2 = \left(\frac{\partial w}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial w}{\partial y}\right)^2 \sigma_y^2 \qquad (19.1)$$

Thus, if we take the function of Section 8(iii),

$$w = x \pm y$$

Then $(\partial w/\partial x) = 1$, and $(\partial w/\partial y) = \pm 1$, giving

$$\sigma_w^2 = \sigma_x^2 + \sigma_y^2 \qquad (19.2)$$

This is just the rule for finding the uncertainty in a sum or difference equation (8.2), expressed in different symbols. It is a surprising result; we might expect that the uncertainty in w would simply be the sum of the uncertainties in x and y. However a simple, if somewhat artificial, example can illustrate that the rule is valid.

Example 19a

Suppose that we have a length $x = 10 \pm 1$mm. Let us imagine that the uncertainty produced by random effects is always exactly ± 1mm, so that whenever we measure x we get either 9 or 11mm with equal probability. Hence $\sigma_x = 1$ mm. Suppose also that we have a similar length, $y = 20 \pm 2$ mm, where the uncertainty behaves in the same way, giving values of 18 and 22 mm. Then $\sigma_y = 2$ mm.

What now is the uncertainty in $w = x + y$? If the uncertainties in x and y are random, then w may take any of the four following values with equal probability:

9 + 18 = 27 mm
11 + 18 = 29 mm
9 + 22 = 31mm
11 + 22 = 33 mm

The mean value of w is thus 30 mm. The mean square deviation σ^2 is found by taking each deviation from the mean (–3, –1, 1, 3), squaring it and multiplying by the probability of its occurrence (which is 1/4).

Thus:

$$\sigma_w^2 = (-3)/4 + (-1)/4 + 1/4 + 3/4 = 5$$

which is the same result as given by equation 19.2

$$\sigma_w^2 = 1^2 + 2^2 = 5$$

Equation 19.1 can be used to obtain the expressions given in Section 8 for the uncertainties in a power and in a product or a quotient. Any other function or combination of variables can be handled in the same way. However the differentiation and subsequent calculation can be tedious, and it is often quicker to calculate the uncertainty in a function directly.

Let us suppose that we have measured a quantity x and have the result $x_0 \pm \Delta x$. We require the corresponding value $y_0 \pm \Delta y$, where y is a function of x, $y = f(x)$.

Let $x_1 = x_0 + \Delta x$ and $x_2 = x_0 - \Delta x$.

Then from $y = f(x)$, we compute the values of y_0, y_1 and y_2 corresponding to x_0, x_1 and x_2. Then, as for x,

$$y_1 = y_0 + \Delta y \text{ and } y_2 = y_0 - \Delta y.$$

Hence Δy can be found.

Example 19b (The function $y = \sin x$)

A beam of laser light passes perpendicularly through a diffraction grating, ruled with 5000 lines per cm (grating spacing $d = 2$ μm). The angle of the first-order diffracted beam is found to be 18° ± 0.5°. Find the wavelength of the light λ and Δλ, assuming that any uncertainty in the value of the grating spacing is very small.

Using $\lambda = d \sin\theta$, we have:

$\theta_0 = 18°$ giving $\lambda_0 = 0.618\ \mu$ m,
$\theta_1 = 18° + 0.5° = 18.5°$, giving $\lambda_1 = 0.634\ \mu$ m,
$\theta_2 = 18° - 0.5° = 17.5°$, giving $\lambda_2 = 0.601\ \mu$ m
Thus $\Delta\lambda = 0.635 - 0.618 = 0.017\ \mu$ m,
also $\Delta\lambda = 0.618 - 0.601 = 0.017\ \mu$ m.
Hence $\Delta\lambda = 0.681 \pm 0.017\ \mu$ m.

Example 19c (The function $y = 1n\ x$)

If $x = (2.5 \pm 0.1) \times 10^6$, what is the uncertainty in $1n\ x$?

$x_0 = 2.5 \times 10^6$, giving $y_0 = 14.732$,
$x_1 = 2.6 \times 10^6$, giving $y_1 = 14.771$,
$x_2 = 2.4 \times 10^6$, giving $y_2 = 14.691$.
Hence $\Delta y = 14.771 - 14.732 = 0.038$,
also $\Delta y = 14.732 - 14.691 = 0.041$.
Thus $y = 14.73 \pm 0.04$.

Note that, because $1n\ x$ is a slowly-changing function, y requires more significant figures than x if accuracy is not to be lost.

20. Uncertainties in experiments on radioactivity

Radioactive decay is a random process. A radioactive source emits its radiation irregularly rather than at a steady rate. If we could record these decay events over many equal intervals of time, we find that there is no single 'true value' for the count. The counts vary randomly from one time interval to the next.

This means that counting radioactive decays is rather more like John's measurements of student heights than Anne's measurements of g. Any particular count is just as valid as any other, telling us how many decays were detected in the time interval. Another similarity is that the reading uncertainties (in timing the counts or in measuring the heights) are usually much smaller than the spread of values obtained (counts, heights).

However, we need a single value, an average count, to represent, for instance, the activity of a radioactive source. If can be shown that the counts obtained in successive time intervals follow what is known as a Poisson distribution. This closely resembles the Normal distribution

(except when the number of counts is very small). In addition, the standard deviation of the distribution is just equal to the square root of the mean count.

Hence, if we have just one count C, then:

(i) the best estimate of the mean of the distribution is just C,
(ii) the uncertainty in this estimate is $\sqrt{C}$.

We can write this more succinctly as:

$$\text{Average count} = C \pm \sqrt{C},$$

which indicates that the average count is likely (with the probability of about 70%) to lie somewhere between the limits $C + \sqrt{C}$ and $C - \sqrt{C}$. We can combine the uncertainty ΔC with other uncertainties in the usual ways.

If the count has been taken over a time t, then the count rate may be written as:

$$\text{Count rate} = \frac{C \pm \sqrt{C}}{t}$$

If measurements are being made using a source of long (compared with the time over which the experiment is being carried out) half-life, then it is better to measure a large number of counts, rather than count for a set period of time. If, say the inverse square law for gamma radiation is being studied, the count-rate falls off quickly with distance from the source. If counts are registered for one minute, the count may be high close to the source, but may fall to, say 100, further away. This count will have an uncertainty of $\sqrt{100}$, i.e. 10, or 10%. If instead, the time to 1000 counts is taken, then the uncertainty in the count is $\sqrt{1000}$, which is 32, or 3%.

Example 20a

(Taken from measurements made in St Andrews after the Chernobyl nuclear reactor accident, 1986)

On 8 May a standard Geiger counter registered a background B of 1623 counts in 63 minutes. A handful of grass was compressed and placed under the counter; a count C of 13825 counts was recorded in 486 minutes. What count rate was being produced by the grass?

For the absolute and fractional uncertainties we have:

$\Delta C = \sqrt{13825} = 120$; $\quad \Delta C/C = 1/\sqrt{C} = 1/120 = 0.85\%$
$\Delta B = \sqrt{1623} = 40$; $\quad \Delta B/B = 1/\sqrt{B} = 1/40 = 2.5\%$

As the fractional uncertainty in the time was much less than these percentages, it can be ignored.

Hence

total count rate = $13825 \pm 120 = 28.45 \pm 0.24$ counts/min
$\qquad\qquad\qquad\qquad\quad 486$

background count rate = $1623 \pm 40 = 25.76 \pm 0.63$ counts/min
$\qquad\qquad\qquad\qquad\qquad\quad 63$

The net count rate is thus 28.45 – 25.76 = 2.69 counts/min. The corresponding uncertainty is given by equation (5.2);

$$\text{uncertainty} = \sqrt{(0.24^2 + 0.63^2)} = 0.67 \text{ counts/min}$$

We round this to one significant figure and obtain the result

$$\text{grass count rate} = 2.7 \pm 0.7 \text{ counts/min}$$

(**Note**: If about 10% of the radiation from the grass sample was entering the counter, the total activity of the sample was about 30 counts/min or 0.5 counts/s (becquerels). This would have been beta-activity only.)

21. Straight-line fitting with calculators

Most calculators will fit a straight line to a set of data points, and of course, all graphics calculators will carry out this process. It is usually described as 'linear regression' and is a form of least squares calculation. The principle of least squares can be seen in the formula for the standard deviation (equation 17.2). The terms $(x_i - m)$ are the deviations of the measured values from the mean m. This mean is the value of x that makes the sum of the squares of the deviations least; hence the term 'least squares'. If the deviations are taken from any other value of x, then the sum of their squares becomes larger.

When a straight line is to be drawn through a set of points, the same idea can be applied. The deviations are now the distances of the points

from the line. Then the 'best line' through he points will be the one for which the sum of the squares of the deviations is a minimum.

In the simplest form of line-fitting, the deviations δy in the y co-ordinates are minimised. This requires us to assume that the uncertainties in the x_i co-ordinates (the x_i error bars) are small enough to be ignored, and that the uncertainties in the y_i co-ordinates (the y_i error bars) are all the same size. These assumptions are not always valid, particularly when y is some function of a measured quantity. Nevertheless they keep the procedures short and simple, and most calculator programs are based on them. They usually produce the gradient m and intercept b of the straight line that minimises the sum $\Sigma(\delta y_i)^2$.

The chief drawback of these programs is that they do not calculate the uncertainties Δm and Δb, but produce instead a 'correlation coefficient'. The idea here is that the program performs a second least squares calculation, this time minimising the x_i deviations, $\Sigma(\delta y_i)^2$. The gradients of the two lines are then compared. Identical lines yield a correlation coefficient of ± 1, the sign depending on the signs of the gradients. If the lines are perpendicular, then the correlation coefficient will be zero. Intermediate alignments produce intermediate values of the correlation coefficient.

This approach is perfectly valid if it is not known if the quantities x and y are related in any functional way. Thus we might study the correlation between Advanced Higher Physics candidates' marks in the investigation and in the written paper. If there is no correlation then the data points will be scattered randomly, and the correlation coefficient will be low. If they are perfectly correlated, then the coefficient will be close to one. But if x and y are already known to be related by a function such as $y = mx + b$, the correlation coefficient adds nothing to our understanding. Its value usually comes out to be greater than 0.9 (except for very poor data) and there is not much to be learned from this.

It is therefore usually much more informative to obtain m and b, and their respective uncertainties, from a graph, than to use the linear regression and correlation programs on a calculator. Some graphics calculators do give the more useful values of the standard deviation of the gradient, and of the intercept.

REFERENCES

Further reading and references

The following publications will be found helpful.

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   This book is written for first-year Physics courses at English universities, and its mathematical approach will be beyond some Advanced Higher pupils. However it covers all aspects of practical physics, including the writing of reports.

2. **Taylor, J R, *An Introduction to Error Analysis*, 1982, Oxford University Press**
   This book is written for American college physics courses. It begins at a very basic level and introduces new concepts gently and gradually. It deals only with uncertainties (and some related topics such as correlation) but covers as much ground as Squires in this area.

3. **Pentz, M and Shott, M, *Handling Experimental Data*, 1988, Open University Press**
   This book is suitable for Advanced Higher students and for first-year university students. The book introduces science students to basic principles and good practice in the collection, recording and evaluation of experimental data and explains clearly how uncertainties and uncertainties should be calculated and expressed. It does not attempt to justify the statistical treatment of uncertainties.

4. ***Physics Guide to Excel*, Higher Still Support Materials, Summer, 1999**

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APPENDIX 1

Appendix 1: Units, prefixes and scientific notation

The Système International d'Unités (SI) is the accepted system of units and should be used at all times. There are seven basic units and two supplementary ones:

Quantity	Name of unit	Symbol
Length	metre	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Temperature	kelvin	K
Luminous intensity	candela	cd
Amount of substance	mole	mol
plane angle	radian	rad
solid angle	steradian	sr

(**Note** that the symbols are always singular, *i.e.* 7 metres is written 7 m)

There are a number of units that can be expressed in terms of the SI units above but which are used frequently by scientists; some of these are listed below.

work, energy, quantity of heat	joule	J
Force	newton	N
Power	watt	W
electric charge	coulomb	C
electric potential difference, electromotive force	volt	V
electric resistance	ohm	Ω
electric capacitance	farad	F
electric field strength	volt per metre	$V\ m^{-1}$
magnetic induction	tesla	T
inductance	henry	H

APPENDIX 2

Appendix 2: Calibration uncertainties in instruments

Manufacturers of scientific measuring instruments know that it is important to state how precisely the scale on the instrument has been calibrated. This table gives typical maximum values for the calibration uncertainties of several common laboratory instruments. The actual calibration uncertainties in particular instruments can be expected to be somewhat less than these.

Wooden metre stick	0.5 mm		
Steel rule	0.1 mm		
Vernier callipers	0.01 mm		
Micrometer	0.002 mm		
Standard masses (chemical balance)	5 mg		
Hg-in-glass thermometer (0°–100°C)	0.5 celsius degree		
Electrical meters			
Analogue	2% of full-scale-deflection		
Digital (3% digit)*	0.5% of reading + 1 digit		
Audio oscillator	5% of full-scale frequency		
Decade resistance box	1% or 0.1% of value		
Resistors	Brown band or code letter	F	1% of value
and	Red	G	2%
Capacitors	Gold	J	5%
	Silver	K	10%
	no	M	20%

* A 4-digit instrument in which the left-hand digit reads 0 or 1 only. Hence the largest figure that can be displayed is 1999.