USE OF CALCULUS IN ADVANCED HIGHER PHYSICS

DIFFERENTIATION

1. OBTAINING VELOCITY FROM DISPLACEMENT TIME GRAPHS AND OBTAINING ACCELERATION FROM VELOCITY TIME GRAPHS.

Let us first understand the concept of ‘rate of change” (with respect to time). For example, you know from Higher Physics that speed is the rate of change of distance and that velocity is the rate of change of displacement and so on.

$$v = \frac{ds}{dt}$$

$$a = \frac{dv}{dt}$$

Think of this as:

$$v = \frac{\Delta s}{\Delta t}$$

(i.e. change in s ÷ change in t)

$$a = \frac{\Delta v}{\Delta t}$$

(i.e. change in v ÷ change in t)

In other words, when we differentiate for a given quantity we obtain the rate of change of that quantity (with respect to time). Simply, this can be found by calculating the gradient of the line.

Example:

From Figure 1, find the acceleration of the yacht.

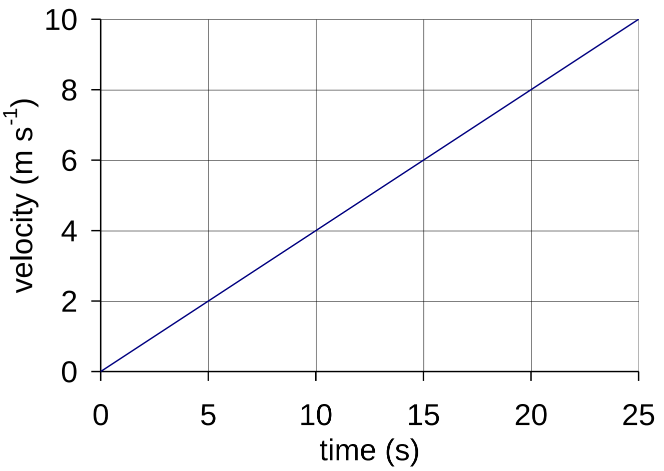


FIGURE 1: ACCELERATION OF A YACHT.

2. Obtaining velocity from acceleration *vs.* time graph (constant acceleration).

From Nat5 you knew that if you want to find displacement you find the area under a velocity time graph. Likewise the velocity can be found from the area under an acceleration time graph.

From Figure 2 calculate the following (assume the vehicle started from rest):

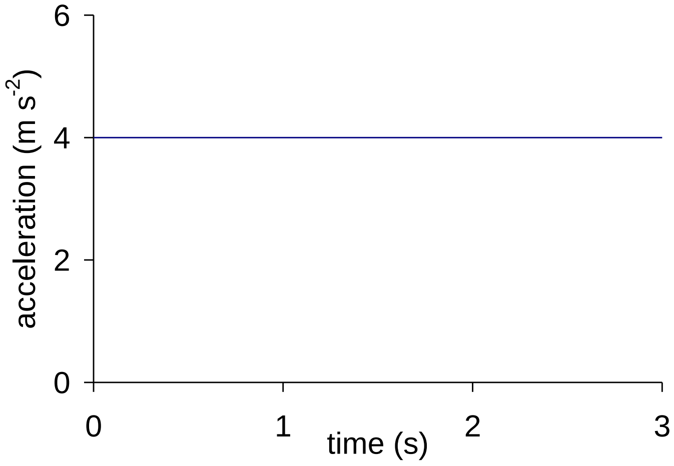


FIGURE 2

a) v at 1 second.

b) Δv from 1 to 2 seconds.

c) v at 2 seconds.

d) Δv from 2 to 3 seconds.

e) v at 3 seconds

3. Obtaining velocity from acceleration *vs*. time graph (stepped accelerations).

It becomes more complicated when the vehicle has changing motion as in the example below, but it is still fairly simple.

From Figure 3 calculate the following (again, the object started at rest):

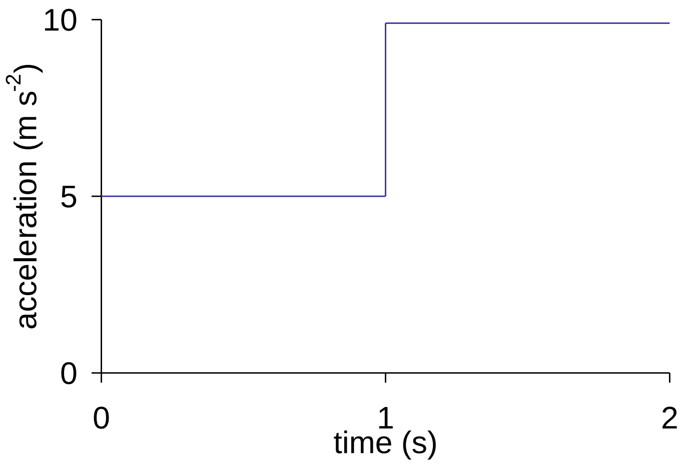


FIGURE 3

a) v at 1 second.

b) Δv from 1 to 2 seconds.

c) v at 2 seconds.

4. Obtaining velocity from acceleration *vs*. time graph (da/dt constant).

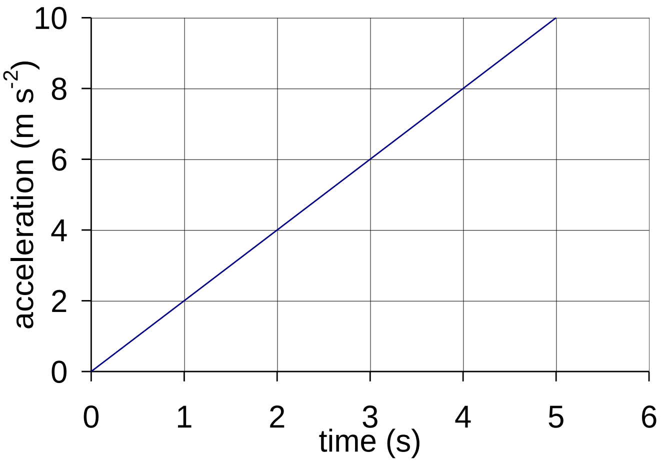


FIGURE 4A

Obtaining the velocity from the previous two graphs was relatively easy, since we could simply add the two well-defined areas together. However, this simple method would be inadequate for a more complex situation such as that described by graph 4a (pretend that you don’t know the formula for the area of a triangle because later this method wont work so we need an alternative!). For this example assume the initial velocity is zero.

In a situation like this, we must estimate the area (and therefore the final velocity) by constructing a series of thin columns. or “elements”, across the time range described (see graph 4b). The thinner the elements, the more accurate we will be (since the value of *a* will not change significantly across each element). We then add the area of each rectangular element together to obtain the final velocity.

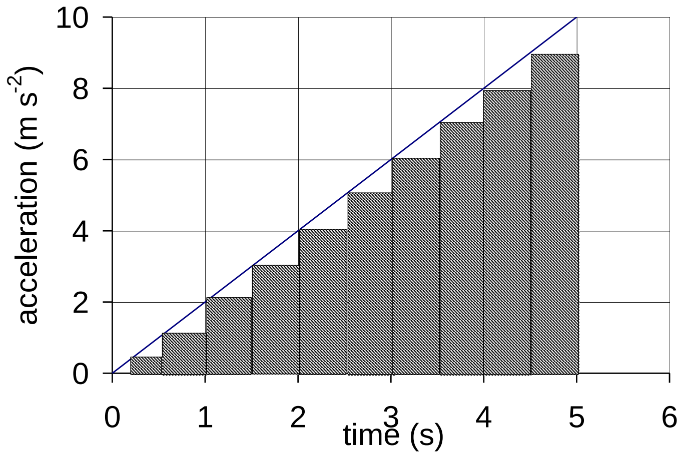


FIGURE 4B

Try this and then compare with the actual area of the triangle method for calculating change in velocity .

This is what we mean by integration.

When the graph is too complex to obtain a simple area, we construct a series of elements and add (or “integrate”) these together. More of this later.

5. Integration

We will now apply this method to a really complex situation.

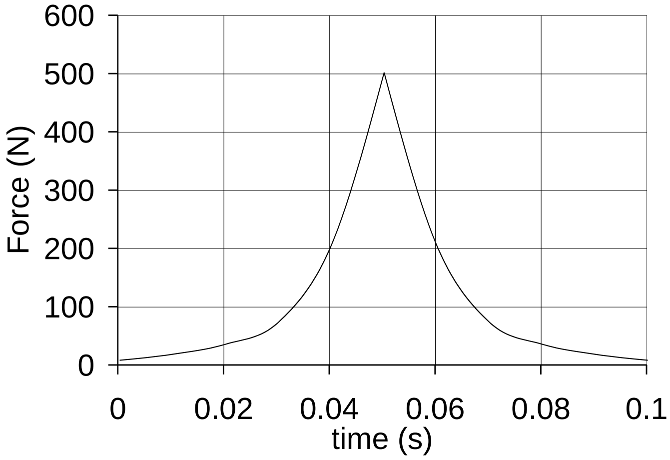


FIGURE 5

Amateur boxers must wear a compulsory head guard (only when they are boxing, obviously) in order to limit the force applied to the head when it is punched. A test using sensors implanted in one such head guard obtained the results shown in Figure 5.

Note that during the contact the force increases to a maximum and then decreases to zero, but not at a constant rate.

In order to calculate the average force we must first obtain the change in momentum then divide by contact time.

Remember that:

Impulse = average Force x contact time

= Δmomentum, which is constant for a given blow.

= area between the line and the time-axis.

Therefore, we can obtain the change in momentum from the area.

In other words, we can obtain Δmomentum by integrating force with respect to time.

Choosing thin elements (across which F does not change significantly), obtain an estimate of the change in momentum.

6. Calculus in Advanced Higher Physics

You will use this skill at various stages of the course, but always in the same way. Integration is particularly useful when we need to establish the product of two variables which are not constant We will return to this later.

Remember:

differentiate (i.e. find gradient. dy/dx)

$\longrightarrow$

displacement **velocity** **acceleration**

$\longleftarrow$

integrate (i.e. find area between line and x-axis)

NUMERICAL QUESTIONS

KINEMATIC RELATIONSHIPS

1. The displacement, s, in metres, of an object after time t in seconds, is given by $s = 90t - 4t^2$.

   (a) Using a calculus method, find an expression for the object's velocity.
   (b) At what time will the velocity be zero?
   (c) Show that the acceleration is constant and state its value.

2. The displacement, s, in metres of a 3 kg mass is given by $s = 8 - 10t + t^2$, where t is the time in seconds.

   (a) Calculate the object's velocity after:
   (i) 2 s
   (ii) 5 s
   (iii) 8 s.

   (b) Calculate the unbalanced force acting on the object after 4 s.
   (c) Comment on the unbalanced force acting on the object during its journey.

3. The displacement, s, of a car is given by the expression $s = 5t + t^2$ metres, where t is in seconds.

   Calculate:
   (a) the velocity of the car when the timing started
   (b) the velocity of the car after 3 seconds
   (c) the acceleration of the car
   (d) the time taken by the car to travel 6 m after the timing started.

4. The displacement, s, of an object is given by the expression $s = 3t^3 + 5t$ metres, where t is in seconds.

   (a) Calculate the displacement, speed and acceleration of the object after 3 seconds.
   (b) Explain why the unbalanced force on the object is not constant.

5. An arrow is fired vertically in the air. The vertical displacement, s, is given by $s = 34.3t - 4.9t^2$ metres, where t is in seconds.

   (a) Find an expression for the velocity of the arrow.
   (b) Calculate the acceleration of the arrow.
   (c) Calculate the initial velocity of the arrow.
   (d) Calculate the maximum height reached by the arrow.

6. The displacement, s, of an object is given by $s = 12 + 15t^2 - 25t^4$ metres, where t is in seconds.

(a) Find expressions for the velocity and acceleration of the object.
(b) Determine the object's initial:
    (i) displacement
    (ii) velocity
    (iii) acceleration.
(c) At what times is the velocity of the object zero?

7. The displacement, s, of a rocket launched from the Earth's surface is given by $s = 2t^3 + 8t^2$ metres for $0 \leq t \leq 30$ seconds.

(a) Calculate the speed of the rocket after 15 seconds.
(b) How far had the rocket travelled in 30 s?
(c) Suggest a reason why the expression for displacement is only valid for the first 30 s.

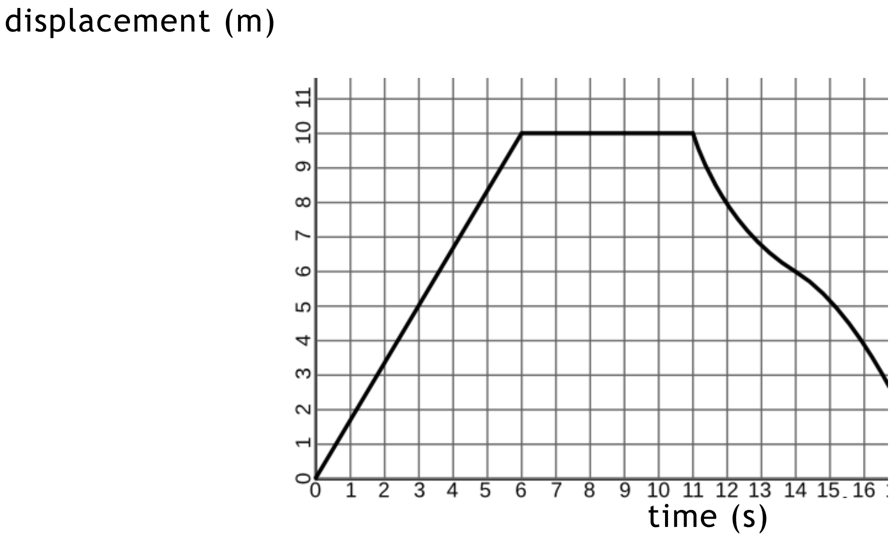
8. A box with a constant acceleration of 4 m s^{-2} slides down a smooth slope. At time $t = 0$ the displacement of the box is 2 m and its velocity is 3 m s^{-1}.

(a) Use a calculus method to show that the velocity v of the box is given the expression $v = 4t + 3$ m s^{-1}.
(b) Show that the displacement of the box is given by $s = 2t^2 + 3t + 2$ metres.

9. The velocity, v, of a moving trolley is given by $v = 6t + 2$ m s^{-1}. The displacement of the trolley is zero at time $t = 0$.

(a) Derive an expression for the displacement of the trolley.
(b) Calculate the acceleration of the trolley.
(c) State the velocity of the trolley at time $t = 0$.

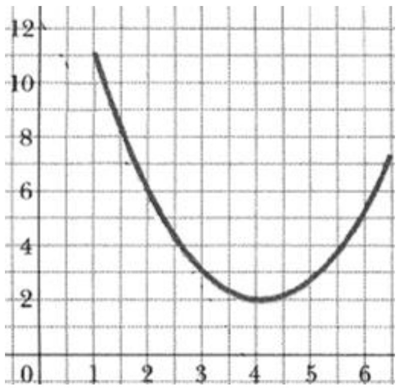
10. The following graph shows the displacement of an object varying with time.



Calculate the velocity of the object at:
(a) 3 s
(b) 8 s
(c) 12 s.

11. The following graph shows how the velocity of an object changes with time.

velocity (m s^{-1})



time (s)

(a) Calculate the acceleration of the object at 2 s.

(b) At what time is the acceleration zero?

(c) Estimate the distance travelled between 2 s and 5 s.

HOMEWORK

Revision Questions for Differentiation and Linear Equations of Motion

Up until now we have only dealt with constant acceleration for linear motion problems. What we need to do is some examples of varying acceleration and using differentiation.

Remember

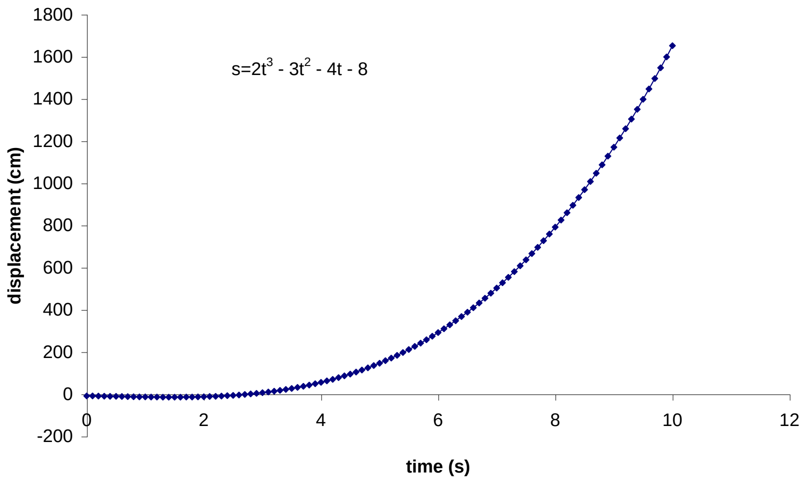
$$velocity = \frac{ds}{dt}$$

instantaneous velocity $= \dfrac{ds}{dt}$ for very small displacements

This is why we use differentiation.

Eg. Suppose an object's displacement is given by the equation

$$s = 2t^3 - 3t^2 - 4t - 8$$



To find v at any time we can differentiate (or take the gradient at time t).

Differentiating gives us:

$$\frac{ds}{dt} = 6t^2 - 6t - 4$$

so find v at t = 3 s $\quad (6\times 3^2) - (6\times 3) - 4 = 32\ \text{cm s}^{-1}$

so find v at t = 2 s $\quad (6\times 2^2) - (6\times 2) - 4 = 8\ \text{cm s}^{-1}$

ACCELERATION IS DEFINED AS RATE OF CHANGE OF THE VELOCITY. IT CAN BE FOUND BY THE GRADIENT OF THE V-T GRAPH. OR DIFFERENTIATE TWICE:

$$\frac{d^2s}{dt^2} = \frac{dv}{dt} = 12t - 6$$

HINTS: TO LOOK FOR CONSTANT SPEED LOOK FOR T;
TO LOOK FOR CONSTANT ACCELERATION LOOK FOR T^2;
to look for varying acceleration look for t^3;

For example: The displacement of an object after time t is:

$$s = t^3 + t^2 - 6$$

Calculate:

a) distance travelled after 2 s and 3 s;
b) speed at t = 2 s and t = 3 s;
c) acceleration at t = 2 s and t = 3 s.

a)

$$t = 2s$$

$$s_2 = 2^3 + 2^2 - 6$$
$$s_2 = 8 + 4 - 6$$
$$s_2 = 6m$$

$$t = 3s$$

$$s_3 = 3^3 + 3^2 - 6$$
$$s_3 = 27 + 9 - 6$$
$$s_3 = 30m$$

b)

$$v = \frac{ds}{dt} = 3t^2 + 2t$$
$$v_2 = 3(2)^2 + 2(2)$$
$$v_2 = 16\,m\,s^{-1}$$

$$v = \frac{ds}{dt} = 3t^2 + 2t$$
$$v_3 = 3(3)^2 + 2(3)$$
$$v_3 = 33\,m\,s^{-1}$$

c)

$$a = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 6t + 2$$
$$a_2 = 6 \times 2 + 2$$
$$a_2 = 14\,m\,s^{-2}$$

$$a = \frac{d^2s}{dt^2} = \frac{dv}{dt} = 6t + 2$$
$$a_3 = 6 \times 3 + 2$$
$$a_3 = 20\,m\,s^{-2}$$

QUESTIONS

1. A ball is thrown vertically upwards with v = 30 m s^{-1}. The height after time t is given as:

$$s = ut + \tfrac{1}{2}at^2$$
$$\therefore h = 30t - 5t^2$$

(for displacement, upwards is positive).

Calculate: i) speed after 3 s and 5 s;

ii) time to reach the Earth;

iii) **time when v=0;**
iv) **greatest height.**

2. **Suppose the motion of a particle is given by**

$v = m + nt^2$ **where m = 10 cm s^{-1} and n = 2 cm s^{-3} (just a constant)**

i) **Find the change in velocity of the particle for the time interval between t_1 = 2 s and t_2 = 5s.**
ii) **Find the average acceleration over this time period.**
iii) **Find the instantaneous acceleration at t_1 = 2s**

3. **Suppose the motion of a particle is described as**

$s = a + bt^2$ **where a = 20 cm and b = 4 cm s^{-2}.**

i) **Find the displacement of the particle for the time interval between t_1 = 2 s and t_2 = 5 s.**
ii) **Find the average velocity during this time interval.**
iii) **Find the instantaneous velocity at t = 2 s.**

DERIVATION OF EQUATIONS OF MOTION.

Acceleration, *a*, is the rate of change of velocity:

Differential equation

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{dt^2}$$

Consider a body accelerating uniformly for a time, *t*. When *t* = 0 the velocity is *u* and is finally *v* at time, *t*.

Then:

$$a = \frac{dv}{dt} \qquad dv = adt$$

Starting and finishing conditions

$$\int_u^v dv = \int_0^t a.dt$$

$$[v]_u^v = [at]_0^t$$

$$v - u = at - 0$$

$$\boxed{v = u + at}$$

And:

$$v = \frac{ds}{dt}$$

$$\int_{s_0}^{s} ds = \int_0^t v.dt$$

$$\int_{s_0}^{s} ds = \int_0^t (u + at)\,dt$$

Subtract the bottom conditions from the top

Raise the power by one and divide by the new power

$$[s]_{s_0}^{s} = \left[ut + \tfrac{1}{2}at^2\right]_0^t$$

$$s - s_0 = ut + \tfrac{1}{2}at^2 - 0$$

$$when\ \ t = 0,\ s = 0\ \therefore s_0 = 0$$

$$\boxed{s = ut + \tfrac{1}{2}at^2}$$

And:

$$a = \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = \frac{dv}{ds} \cdot v \qquad \therefore a \cdot ds = v \cdot dv$$

$$\int_0^s a \cdot ds = \int_u^v v \cdot dv$$

$$[as]_0^s = \left[\tfrac{1}{2}v^2\right]_u^v$$

$$as - 0 = \tfrac{1}{2}v^2 - \tfrac{1}{2}u^2$$

$$v^2 - u^2 = 2as$$

$$\boxed{v^2 = u^2 + 2as}$$

This is a neat trick which allows us to eliminate time from the differential equation.

CONSTANT VELOCITY

$$s = 4t + 8$$

$$\frac{ds}{dt} = 4$$

CONSTANT ACCELERATION

$$s = 2t^2 + 7$$

$$\frac{ds}{dt} = 4t$$

$$\frac{d^2s}{dt^2} = 4$$

REMEMBER:

$$v = \frac{ds}{dt}$$

$$a = \frac{dv}{dt}$$

$$a = \frac{d^2s}{dt^2}$$

BASICS OF MOTION CALCULATION

$$x = a + bt^2$$

eg $x = 20 + 4t^2$

$$\frac{dx}{dt} = 8t$$

if t = 2 s

$$\frac{dx}{dt} = 16$$

TO FIND *v* AT ANY TIME SUBSTITUTE IN *t*

constant velocity look for$\propto$ t

constant acceleration look for $\propto\propto t^2$

non uniform acceleration look for $\propto t^3$ or greater.

2002 Q1

1. (*a*) An object moves with constant acceleration a.

At time $t = 0$ its displacement s is zero.

The velocity v of the object is given by $v = u + at$.

Derive the equation

$$s = ut + \tfrac{1}{2}at^2$$

where the symbols have their usual meanings.

2012 Q2

2. (*a*) The acceleration of a particle moving in a straight line is given by

$$a = \frac{dv}{dt}$$

where the symbols have their usual meaning.

(i) Show, by integration, that when a is constant

$$v = u + at.$$

(ii) Show that when a is constant

$$v^2 = u^2 + 2as.$$

2007 Q1

1. (*a*) A particle has displacement $s = 0$ at time $t = 0$ and moves with constant acceleration a.

The velocity of the object is given by the equation $v = u + at$, where the symbols have their usual meanings.

Using calculus, derive an equation for the displacement s of the object as a function of time t.

2009 Q1

1. Figure 1A shows a space shuttle shortly after take-off.

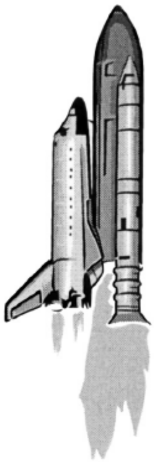


Figure 1A

(*a*) Immediately after take off, the vertical displacement of the shuttle for part of its journey can be described using the equation

$$s = 3{\cdot}1t^2 + 4{\cdot}1t.$$

(i) Find, by differentiation, the equation for the vertical velocity of the shuttle.

(ii) At what time will the vertical velocity be $72\,\text{m s}^{-1}$?

(iii) Calculate the vertical linear acceleration during this time.

2014

1. The acceleration of a particle moving in a straight line is described by the expression

$$a = 1{\cdot}2t.$$

At time, $t = 0\,\text{s}$ the displacement of the particle is $0\,\text{m}$ and its velocity is $1{\cdot}4\,\text{m s}^{-1}$.

(*a*) Show that the velocity of the particle at time t is given by the expression

$$v = 0{\cdot}6t^2 + 1{\cdot}4.$$

(*b*) Calculate the displacement of the particle when its velocity is $3{\cdot}8\,\text{m s}^{-1}$.

2017 Q1

1. An athlete competes in a one hundred metre race on a flat track, as shown in Figure 1A.

Figure 1A

Starting from rest, the athlete's speed for the first 3·10 seconds of the race can be modelled using the relationship

$$v = 0\cdot 4t^2 + 2t$$

where the symbols have their usual meaning.

According to this model:

(a) determine the speed of the athlete at $t = 3\cdot 10$ s;

(b) determine, using **calculus** methods, the distance travelled by the athlete in this time.

Numerical answers	
Kinematic relationships	
	6. (a) $v = 30t - 100t^3$; $a = 30 - 300t^2$
1. (a) $90 - 8t$	(b) (i) 12 m
(b) 11.25 s	(ii) 0
(c) -8 m s^{-2}	(iii) 30 m s^{-2}
	(b) 0 s and 0.55 s
2. (a) (i) -6 m s^{-1}	
(ii) 0	7. (a) 1590 m s^{-1}
(iii) 6 m s^{-1}	(b) 61200 m
(b) 6 N	(c) The rocket runs out of fuel.
(c) Constant	
	9. (a) $s = 3t^2 + 2t$
3. (a) 5 m s^{-1}	(b) 6 m s^{-2}
(b) 11 m s^{-1}	(c) 2 m s^{-1}
(c) 2 m s^{-2}	
(d) 1 s	10. (a) 1.67 m s^{-1}
	(b) Zero
4. (a) 96 m; 86 m s^{-1}; 54 m s^{-2}	(c) -1.25 m s^{-1} (gradient of tangent to slope at 12
(c) a depends on time t	
	11. (a) -3.2 m s^{-2}
5. (a) $v = 34.3 - 9.8t$	(b) 4s (or 4.1 s)
(b) -9.8 m s^{-2}	(c) Approx. 8.75 m (17½ boxes)
(c) 34.3 m s^{-1}	
(d) 60 m	

2002 Q1:

1. (a)
$$\frac{ds}{dt} = v \therefore ds = v.dt$$
$$\int ds = \int (u + at).dt$$
$$s = ut + \tfrac{1}{2}at^2 + c$$
$$\text{at } t = 0,\ c = 0$$
$$\therefore s = ut + \tfrac{1}{2}at^2$$

1997 Q1

$$\theta = \underset{(\frac{1}{2})}{\int \omega\, dt} = \int (\underset{(\frac{1}{2})}{\omega_o} + \alpha t)\, dt$$

$$= \omega_o t + \underset{(\frac{1}{2})}{\tfrac{1}{2}\alpha t^2} + \underset{(\frac{1}{2})}{k}$$

When $t = 0,\ \underset{(\frac{1}{2})}{\theta = 0}$ hence $\underset{(\frac{1}{2})}{k = 0}$

Therefore $\theta = \omega_o t + \frac{1}{2}\alpha t^2$

$$v = u + at$$
$$v^2 = (u + at)^2$$
$$v^2 = (u + at)(u + at)$$
$$v^2 = u^2 + uat + uat + a^2t^2$$
$$v^2 = u^2 + 2uat + a^2t^2$$
$$v^2 = u^2 + 2uat + \tfrac{2}{2}a^2t^2$$
$$v^2 = u^2 + 2a(ut + \tfrac{1}{2}at^2)$$
$$BUT\ \ s = (ut + \tfrac{1}{2}at^2)$$

so by substitution

$\underline{v^2 = u^2 + 2as}$ ---------------3.

2012 AH

$a = \dfrac{dv}{dt}$ $\int dv = \int a.dt$ or $\int \dfrac{dv}{dt}dt = \int a.dt$ $v = at + c$ at t = 0, c = u Must be specific with respect to time $\underline{\underline{v = u + at}}$		$a = \dfrac{dv}{dt}$ $\int_u^v dv = \int_0^t a.dt$ ½ for integrals, ½ for limits need both before can progress $[v]_u^v = [at]_0^t$ $v - u = at\ (-0)$ $v = u + at$
$v^2 = (u + at)(u + at)$ $v^2 = u^2 + 2uat + a^2t^2$		Starting with s=ut+ ½at^2

$v^2 = u^2 + 2a(ut + \frac{1}{2}at^2)$ (½)	½ for substitution for t ½ for manipulation Check second line both *a* and *t* are squared.
$\underline{\underline{v^2 = u^2 + 2as}}$	

$$a = \frac{dv}{dt}$$

$$\int dv = \int a.dt \quad \text{or} \quad \int \frac{dv}{dt} dt = \int a.dt \qquad (½)$$

$$v = at + c \qquad (½)$$

at $t = 0$, $c = u$ (1)

Must be specific with respect to time

$v = u + at$ SHOW ME

$$\frac{ds}{dt} = v$$

$$\int ds = \int (u + at) \cdot dt \qquad (½)$$

$$s = ut + ½\,at^2 + c \qquad (½)$$

$$\text{at } t = 0,\ s = 0,\ \text{so } c = 0 \qquad (½)$$

$$\therefore s = ut + ½\,at^2 \qquad (½)$$

No c or limits $\Rightarrow$ max (½)
If first line is W.P. – (0)
eg WP = $\int s\ ,\ \int \frac{ds}{dt}\ ,\ \int v$

OR

$$\int_o^s ds = \int_o^t (u + at)\,dt$$

(½) equation
(½) limits

$$[s]_o^s = \left[ut + ½\,at^2\right]_o^t \quad (½)$$

$$s = ut + ½\,at^2 \quad (½)$$

$$\frac{ds}{dt} = v \therefore ds = v.dt \qquad (\tfrac{1}{2})$$

$$\int ds = \int (u + at).dt \qquad (\tfrac{1}{2})$$

$$s = ut + \tfrac{1}{2}at^2 + c \qquad (\tfrac{1}{2})$$

$$\text{at } t = 0,\ c = 0 \qquad (\tfrac{1}{2})$$

$$\therefore s = ut + \tfrac{1}{2}at^2$$

2014

$a = \frac{dv}{dt} = 1 \times 2t$ (½) $\int \frac{dv}{dt} \cdot dt = \int 1 \cdot 2t \cdot dt$ (½) $v = 0 \cdot 6t^2 + c$ (½) at $t = 0$, $v = 1 \cdot 4$, $c = 1 \cdot 4$ (½) $v = 0 \cdot 6t^2 + 1 \cdot 4$	**2**	Alternative for step 1 $a = 1 \cdot 2t$ $\int a.dt = \int 1 \cdot 2t \cdot dt$ Alternative for step 2 $\int_{1 \cdot 4}^{v} dv = \int_{0}^{t} 1 \cdot 2t \cdot dt$ $v - 1 \cdot 4 = 0 \cdot 6t^2$

2017

Question			Answer	Max Mark	Additional Guidance
1.	(a)		$v = 0{\cdot}4t^2 + 2t$ $v = (0{\cdot}4 \times 3{\cdot}10^2) + (2 \times 3{\cdot}10)$ 1 $v = 10{\cdot}0\,\mathrm{m\,s^{-1}}$ 1 Accept: 10, 10·04, 10·044	**2**	
	(b)		$s = \int (0 \cdot 4t^2 + 2t).dt$ $s = \frac{0 \cdot 4}{3} t^3 + t^2 (+c)$ 1 $s = 0$ when $t = 0$, $c = 0$ $s = \frac{0 \cdot 4}{3} \times (3 \cdot 10)^3 + 3 \cdot 10^2$ 1 $s = 13 \cdot 6\,\mathrm{m}$ 1 Accept: 14, 13·58, 13·582	**3**	Solution with limits also acceptable. $s = \int_0^{3.10} (0 \cdot 4t^2 + 2t).dt$ $s = \left[\frac{0 \cdot 4 \times t^3}{3} + t^2\right]_{(0)}^{(3.10)}$ 1 $s = \left(\frac{0 \cdot 4 \times 3 \cdot 10^3}{3} + 3 \cdot 10^2\right) - 0$ 1 $s = 13 \cdot 6\,\mathrm{m}$ 1 (s=13·5 intermediate rounding. max 2)